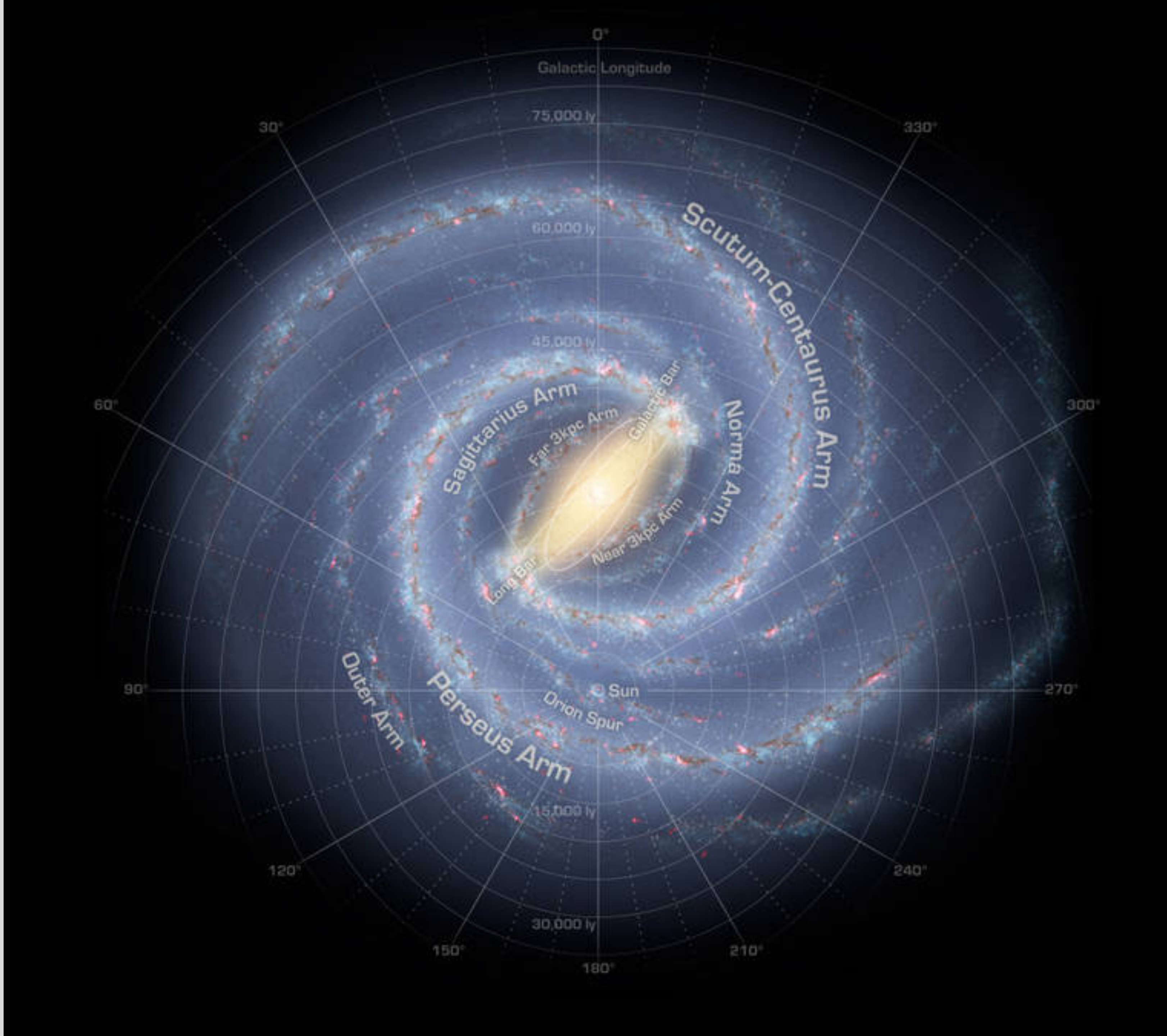
Milky Way Rotation, Orbits, and Epicycles



Milky Way Rotation Speed

Important: for this discussion, V refers to the rotation speed, not the speed relative to the LSR. And also assume stars are on circular orbits.

Estimate of $V(R_0)$ from kinematics of globular clusters and halo stars: ~ 200 km/s. But how can we map this as a function of radius?

Think about the observed radial velocity of a star, which is a combination of our motion and its motion:

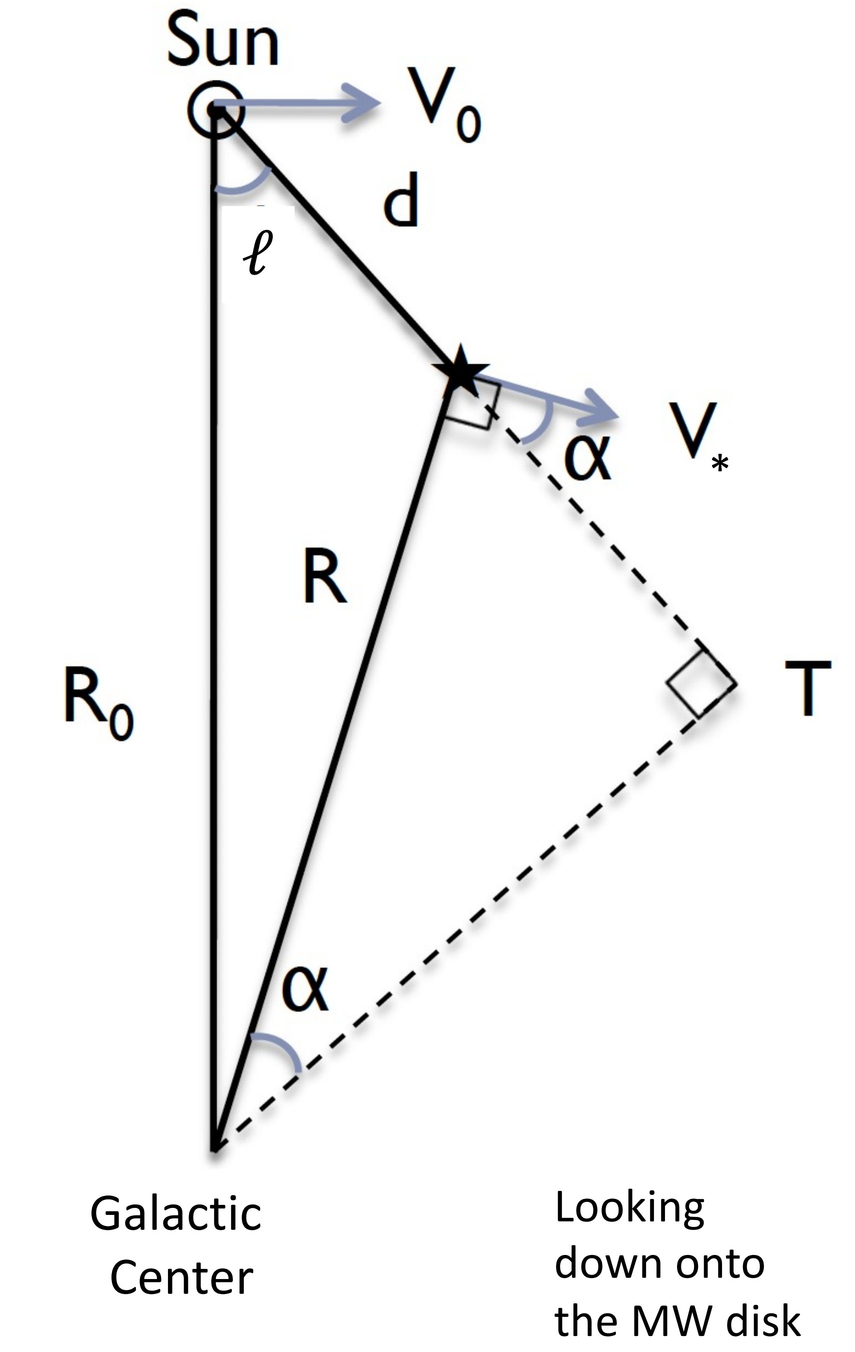
$$v_r = V_* \cos\alpha - V_o \sin\ell$$

If we define the angular velocity as $\Omega = V/R$ and use the law of sines, this turns into

$$v_r = (\Omega_* - \Omega_0)\, R_0 \sin\ell$$

We can make similar arguments about the tangential velocity

$$v_T = (\Omega_* - \Omega_0) R_0 \cos\ell - \Omega_* d$$



Milky Way Rotation Speed

Focus now on radial velocities: $v_r = (\Omega_* - \Omega_0)\, R_0 \sin \ell$

Nominally, since $\Omega = V/R$, we need to know distances to get R's.

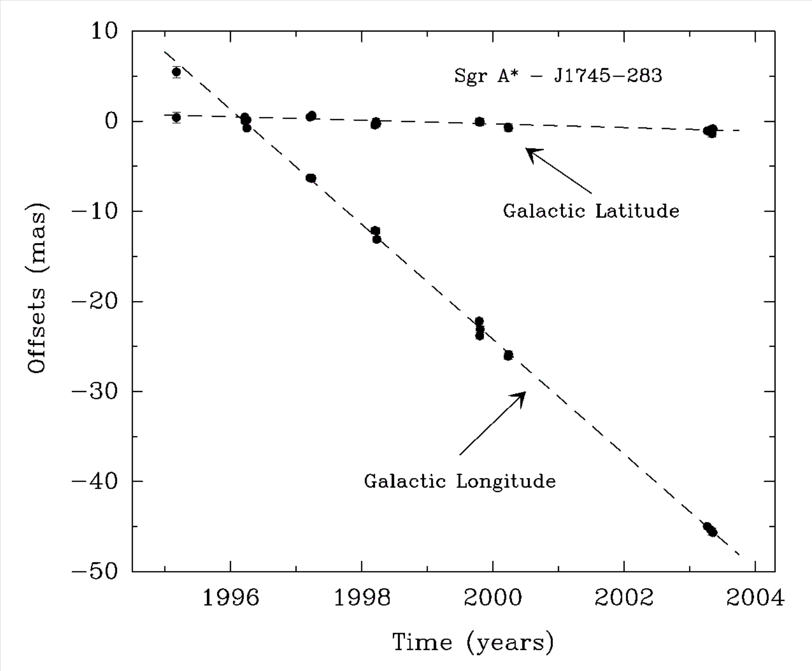
But notice that along that line of sight, the maximum velocity measured will be at the tangent point T. At that point $d = R_0 \cos \ell$.

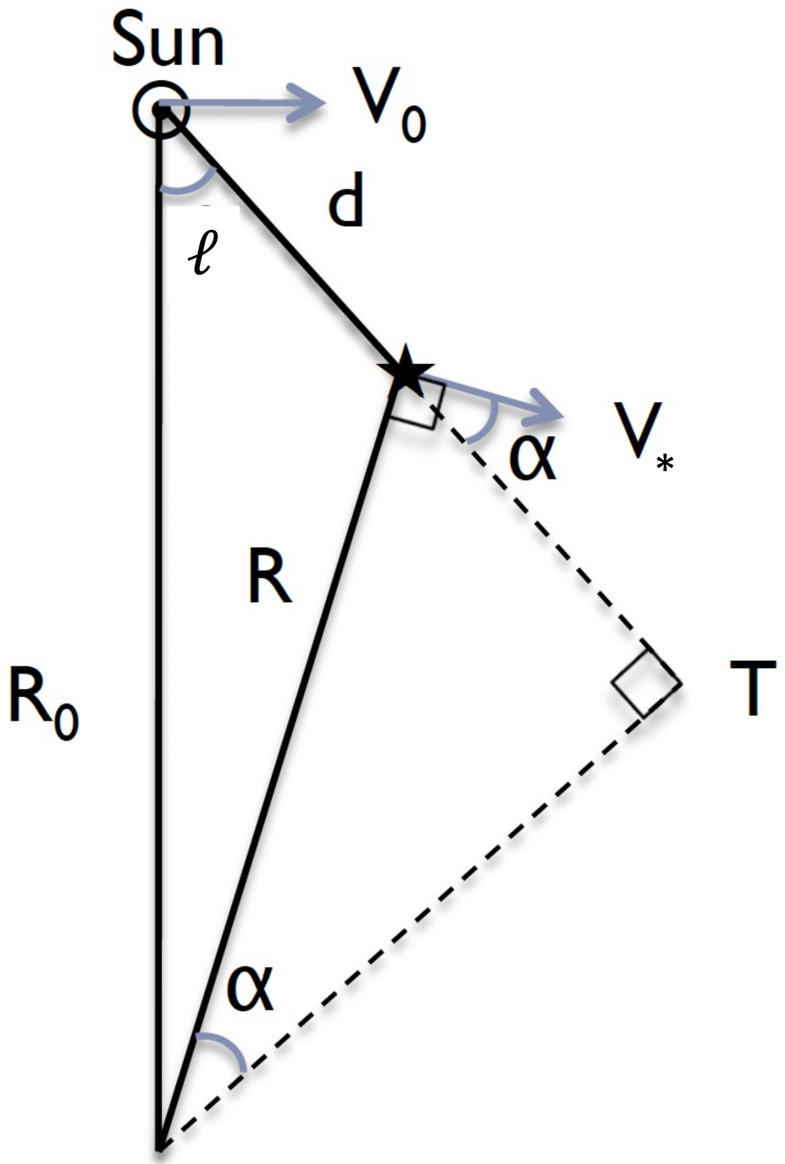
We also need to know $\Omega_0 \equiv V_0/R_0$. Can get this by knowing R_0 and V_0, or (now) by measuring the proper motion of the Sgr A*, the radio source at the Galactic Center.

Sgr A* appears to move because we are moving. Its angular motion on the sky is our angular motion through the Galaxy.

$\Omega_0 = 29.5$ km/s/kpc

Reid & Brunthaler 04





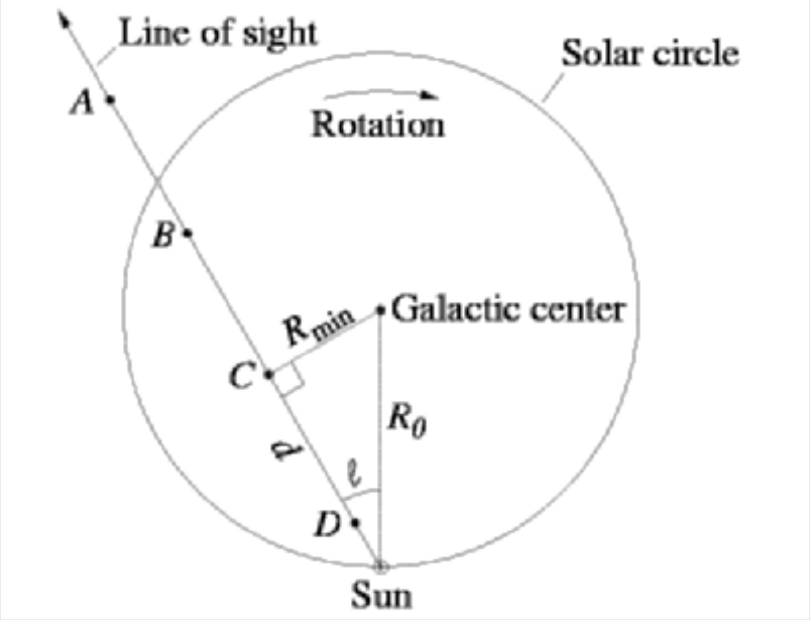
Galactic Center

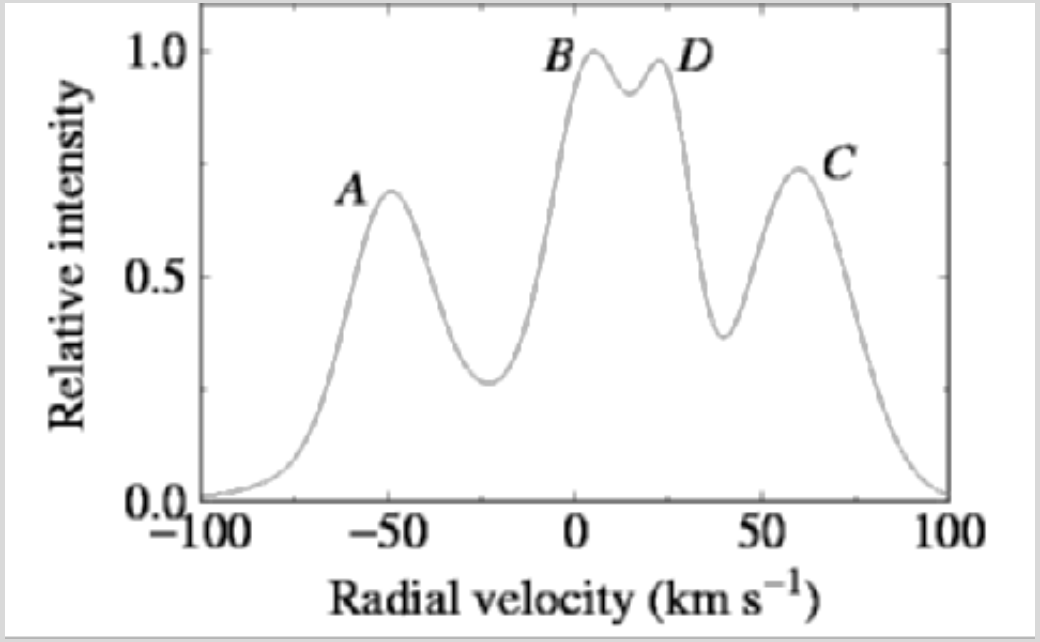
Looking down onto the MW disk

Milky Way Rotation Speed

Want to map velocities of objects in the disk moving on circular orbits. What kinds of objects are these? *gas clouds!*

21-cm HI emission: no extinction at radio wavelengths. Map the HI velocities as a function of Galactic longitude, look for maximum velocity. Imagine gas clouds strung out along some line of sight, and the velocities you measure:





The velocity of cloud C should be the circular speed at $R_{min} = R_0 \sin \ell$.

Works well inside the solar circle: R < R_0. Beyond that, there is no tangent point and actual distances are needed. Use other tracers of young stars: Cepheids, HII regions, etc.

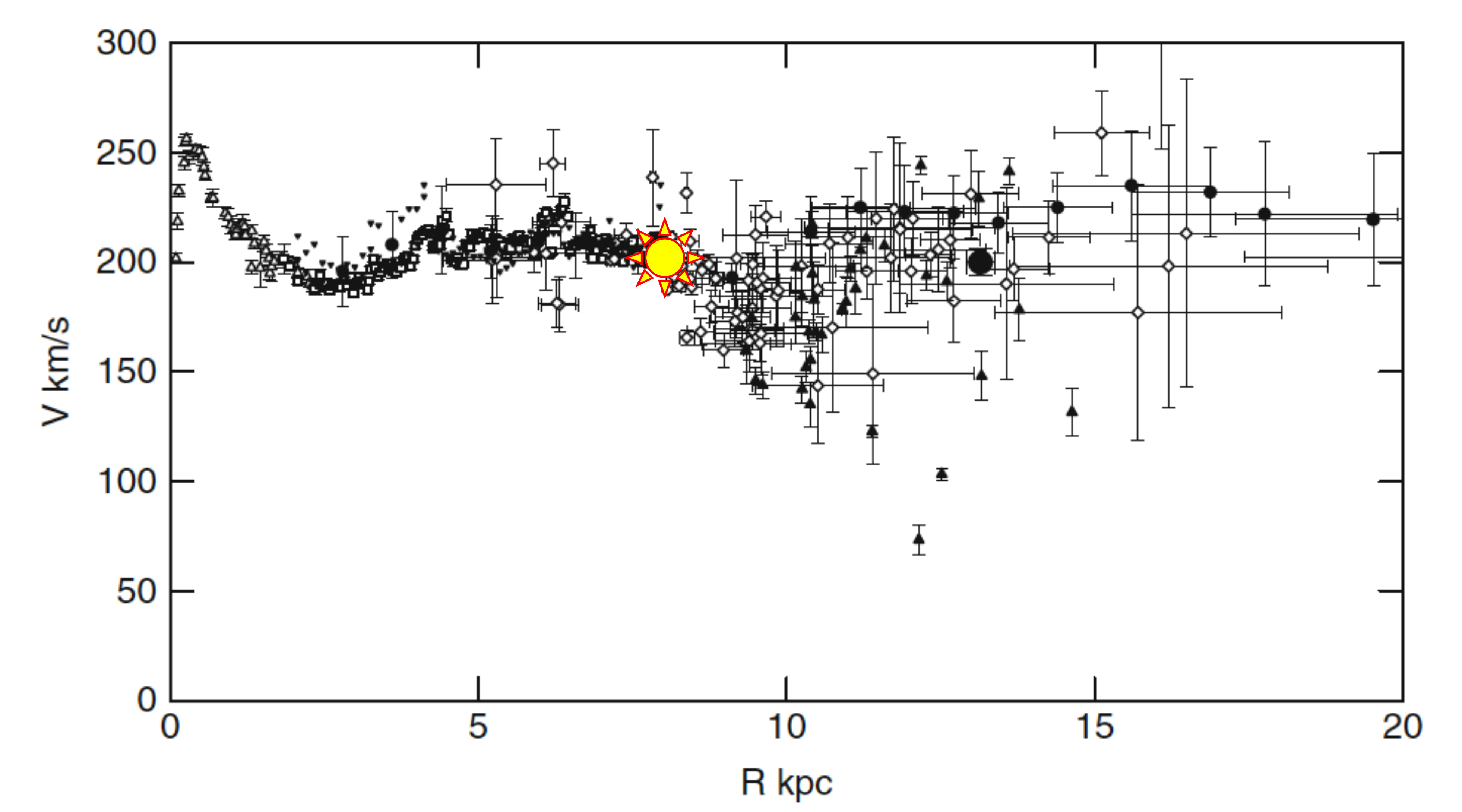
Milky Way Rotation Curve

Sofue 09

IAU "standard":

R_0 = 8.5 kpc
$V_c(R_0)$ = 220 km/s

(but these numbers have been updated….)

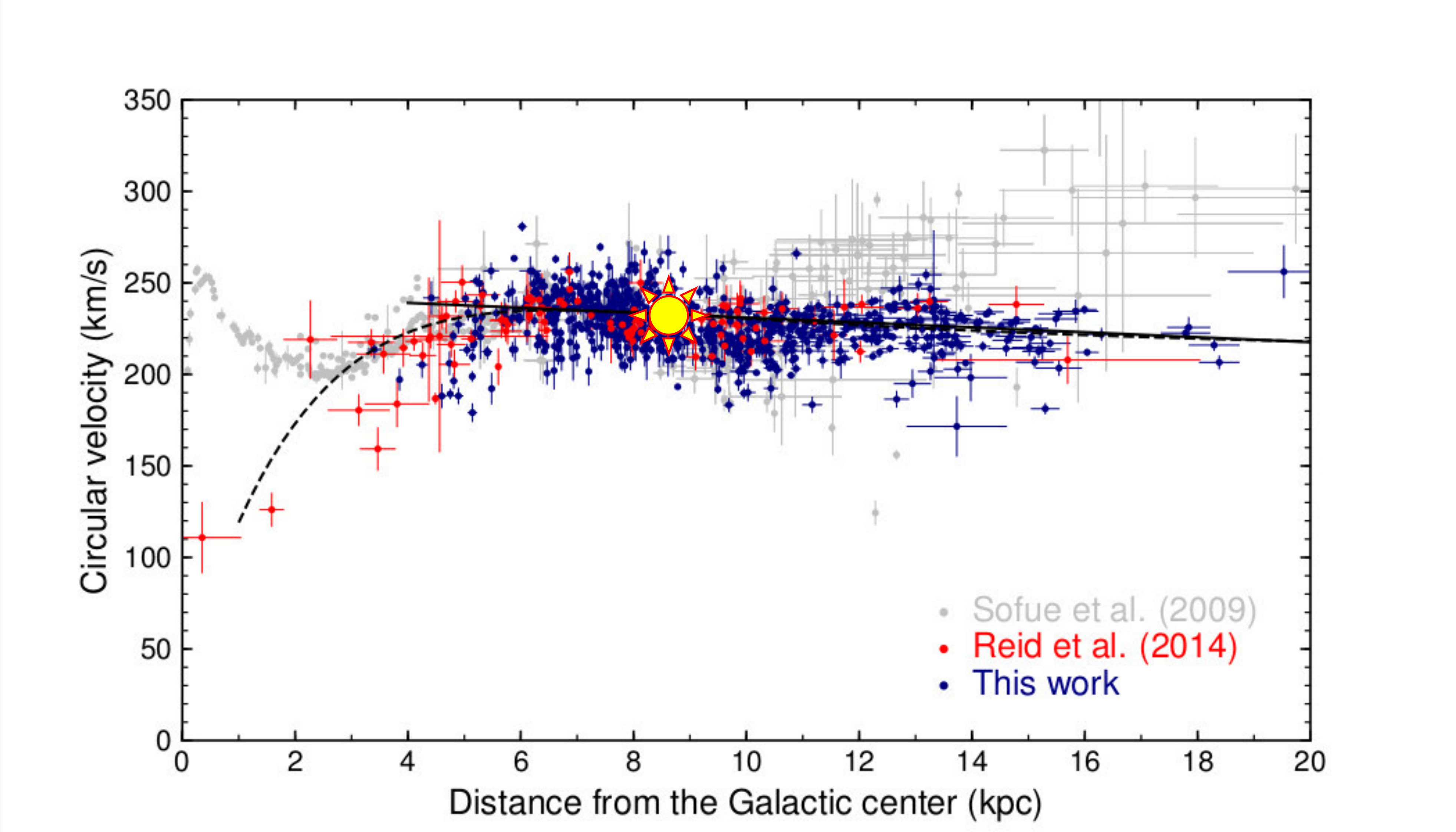


Milky Way Rotation Curve

Mroz+ 19

Gaia Cepheid data
plus
updated R_0

$V_c(R_0)$ = 234 km/s



Rotation Curve, Mass Density, Potential *(a review of PHYS 1)*

A spherical density distribution $\rho(r)$ leads to a interior mass

$$M(<r) = 4\pi \int_0^r \rho(r) r^2 dr$$

which leads to a gravitational potential given by

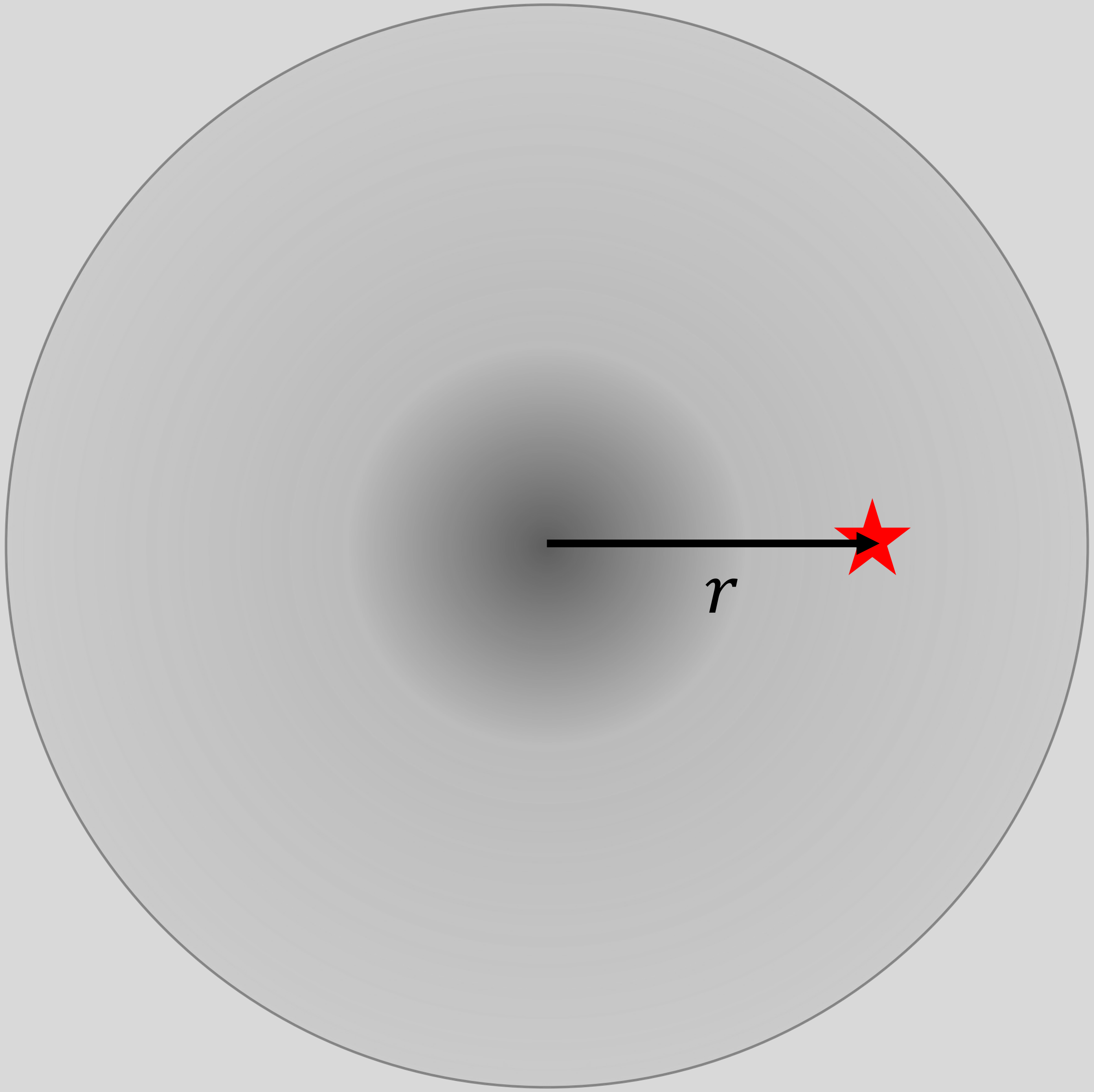
$$\phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r r^2 \rho(r) dr + \int_r^\infty r\rho(r) dr \right]$$

The force felt by a particle at distance r is given by

$$\vec{F} = m\vec{a} = -m\nabla\phi\hat{r} = -\frac{GM(<r)m}{r^2}\hat{r}$$

which leads to a circular speed given by

$$V_c^2 = r\frac{\partial\phi}{\partial r} = \frac{GM(<r)}{r}$$



Rotation Curve, Mass Density, Potential

Disks are not spherical, they are flattened.

Disk surface density: $\Sigma(R) = \Sigma_0 e^{-R/h}$

Integrate to get mass interior:

$$M(R) = 2\pi \int_0^R \Sigma(R) R dr = 2\pi\Sigma_0 h^2 \left(1 - e^{-R/h}\left(1 + \frac{R}{h}\right)\right)$$

Solve for in-plane potential:

$$\phi(R)_{z=0} = -\pi G \Sigma_0 R\big(I_0(y)K_0(y) - I_1(y)K_1(y)\big)$$

where $y = r/2h$ and I_0, K_0, I_1, K_1 are Bessel functions.

Solve for circular velocity:

$$V_c^2 = r\frac{\partial\phi}{\partial r} = 4\pi G\Sigma_0 h y^2\big(I_0(y)K_0(y) - I_1(y)K_1(y)\big)$$

This is the solution for a razor-thin disk. Disks have thickness, describe as oblateness $q = h_z/h_R$

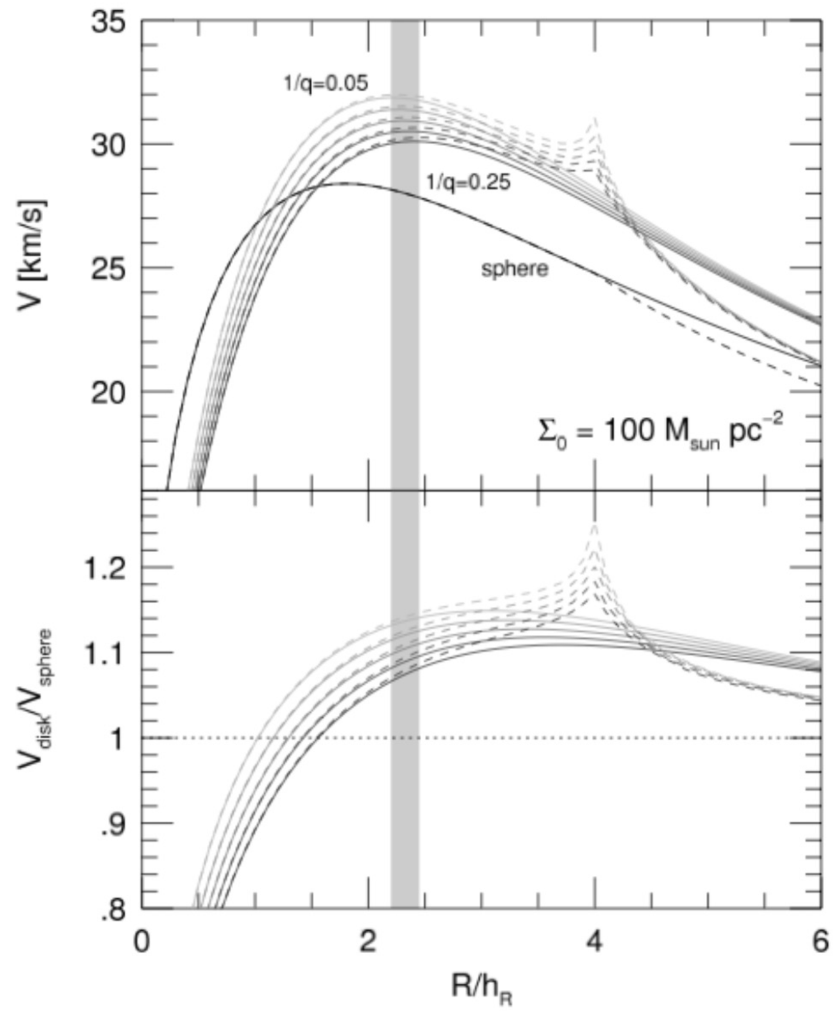
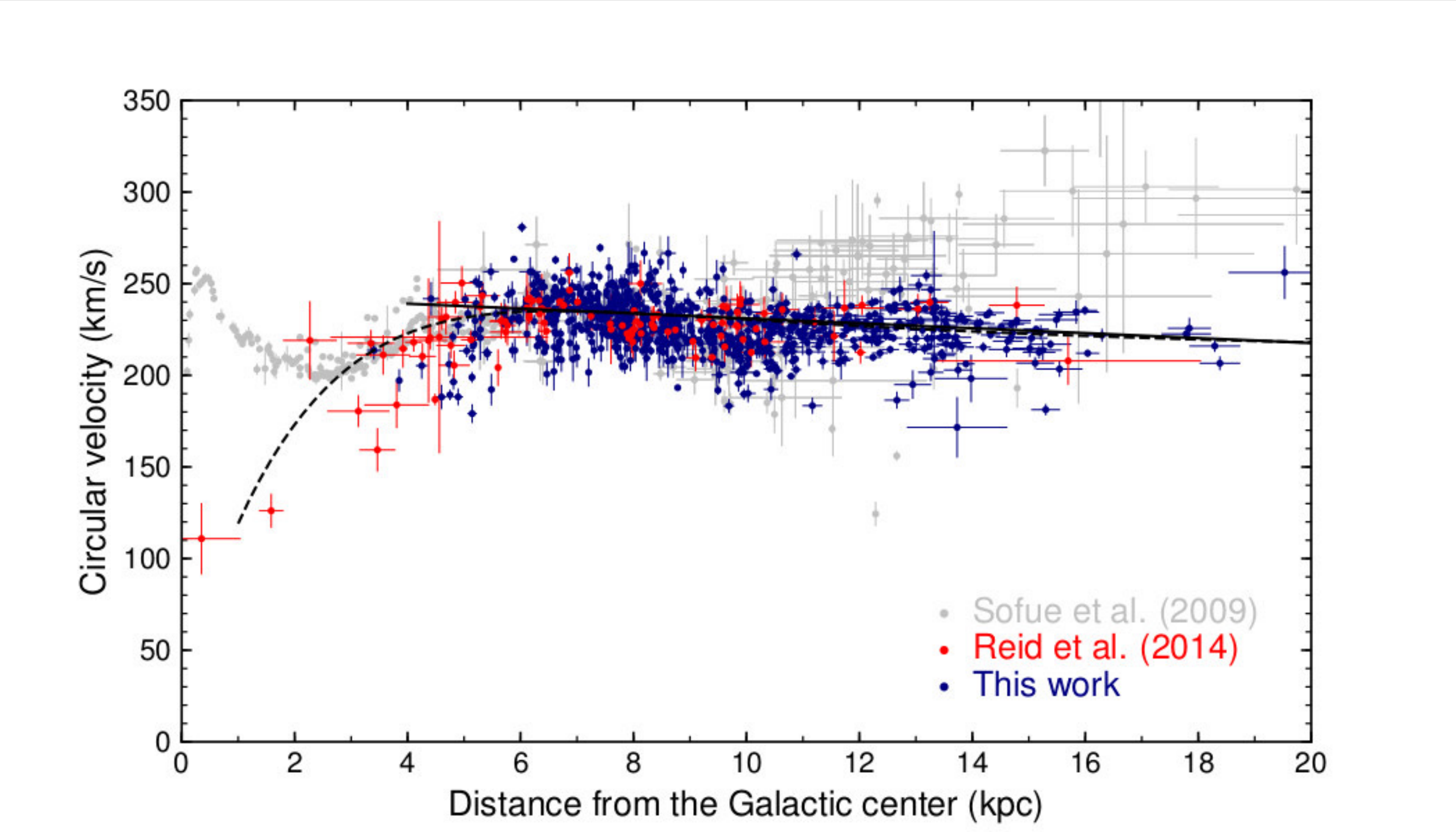


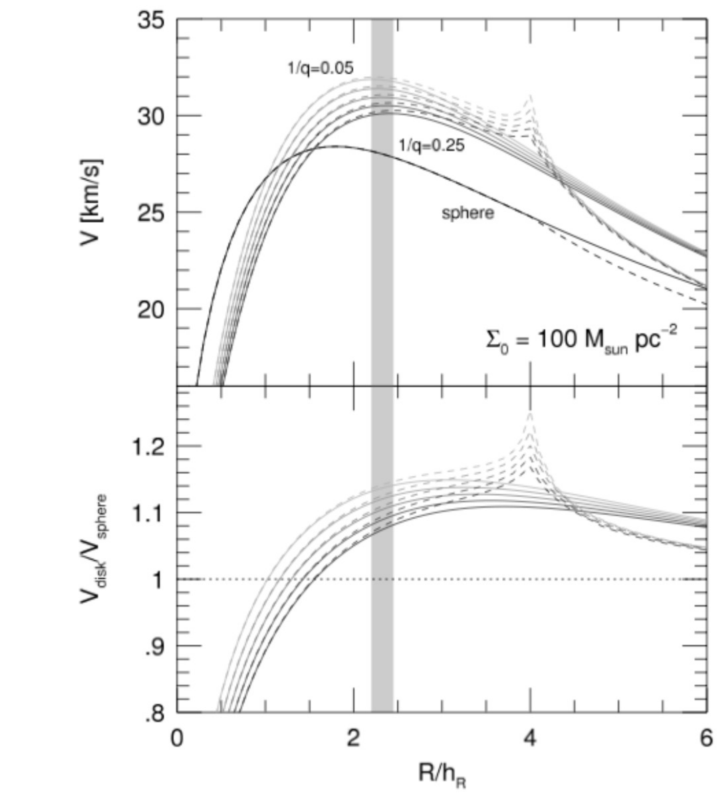
Fig. 17.— Rotation speed of an exponential disk with central mass surface density of 100 $\mathcal{M}_\odot$ pc^{-2} and oblateness $0.05 < q < 0.25$ versus radius normalized by scale-length, compared to a spherical density distribution with the same enclosed mass. Bottom panel shows the ratio of spherical to disk velocities. Dashed and solid lines show disks truncated at R/h_R=4 and 10, respectively. The radial range where these disks have peak velocities is shaded in gray.

BUT THE POINT IS…..



⇐ THIS…

…IS NOT THIS. ⇓



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We need to add an extended halo of “dark matter”: more mass at large radius boosts the rotational speed of the outer disk.

(Or we need to change our understanding of gravity….)

Milky Way Rotation: Differential rotation

The rotation curve of the Milky Way (and other galaxies) is not a "solid body" rotation curve ($V(R) \propto R$). This means objects at different radii will orbit at different angular speeds:

Circular speed: $V(R)$ in km/s.

Angular speed: $\Omega(R) = V(R)/R$ (typically expressed in km/s/kpc)

But note that the units of angular speed are essentially inverse time, so it is basically an ***orbital frequency****.*

Orbital time: $T_{orb}(R) = 2\pi R/V(R) = 2\pi/\Omega(R)$

Since stars at different radii have different angular speeds and orbital times, this introduces shear in the Galactic disk.

Relating gradients: If $\Omega = V/R = VR^{-1}$, then by the product rule for differentiation:

$$\frac{d\Omega}{dR} = \frac{1}{R}\frac{dV}{dR} - \frac{V}{R^2} = \frac{1}{R}\left(\frac{dV}{dR} - \frac{V}{R}\right)$$

Milky Way Rotation: Differential rotation and the Oort Constants

For stars near the Sun, we can make linear approximations to solve for expressions describing shear and vorticity of stellar velocity field.

Expand the angular velocity curve as a Taylor series:	$\Omega(R) = \Omega_0(R_0) + \left.\frac{d\Omega}{dR}\right\|_{R_0} (R - R_0)$ + …

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So to first order:	$\Omega(R) - \Omega_0(R_0) \cong \left.\frac{d\Omega}{dR}\right\|_{R_0} (R - R_0)$

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Take the expression for observed radial velocity:	$v_r = (\Omega_* - \Omega_0)\, R_0 \sin l$

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insert expression for $\Omega(R) - \Omega_0(R_0)$ and expand $\frac{d\Omega}{dR}$:	$v_r \cong \left[\left.\frac{dV}{dR}\right\|_{R_0} - \frac{V_0}{R_0}\right] (R - R_0) \sin l$

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If $d \ll R_0$ we can use the small angle approximation:	$(R - R_0) \approx -d \cos l$
And use a trig identity $(2\cos l \sin l = \sin 2l)$ to get to:	$v_r \cong Ad \sin 2l$ where $\mathrm{A} = -\frac{1}{2}\left[\left.\frac{dV}{dR}\right\|_{R_0} - \frac{V_0}{R_0}\right]$

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A similar analysis on the tangential velocities gives:	$v_T \cong Ad \cos 2l + Bd$ where $B = -\frac{1}{2}\left[\left.\frac{dV}{dR}\right\|_{R_0} + \frac{V_0}{R_0}\right]$

The expressions for A and B were first worked out by Jan Oort in the 1920s and are known as the **Oort Constants**.

Milky Way Rotation: Differential rotation and the Oort Constants

Oort A measures shear, the deviation from rigid rotation.

In rigid rotation, $V = \left(\frac{V_0}{R_0}\right) R$ so A=0.

Oort B measures vorticity of the local velocity field, the tendency for objects to circulate around a position.

They also can be expressed in terms of the velocity curve:

Sun's Angular Velocity	$\Omega_0 = \frac{V_0}{R_0} = A - B$
Circular Velocity at R_0 (i.e., the LSR)	$V_0 = R_0(A - B)$
Circular Velocity Gradient	$\left.\frac{dV}{dR}\right\vert_{R_0} = -(A + B)$
Velocity Dispersion Ellipsoid	$\frac{-B}{A - B} = \frac{\sigma_V^2}{\sigma_U^2}$

$$\mathrm{A} = -\frac{1}{2}\left[\left.\frac{dV}{dR}\right|_{R_0} - \frac{V_0}{R_0}\right]$$

$$\mathrm{B} = -\frac{1}{2}\left[\left.\frac{dV}{dR}\right|_{R_0} + \frac{V_0}{R_0}\right]$$

Bovy 17:

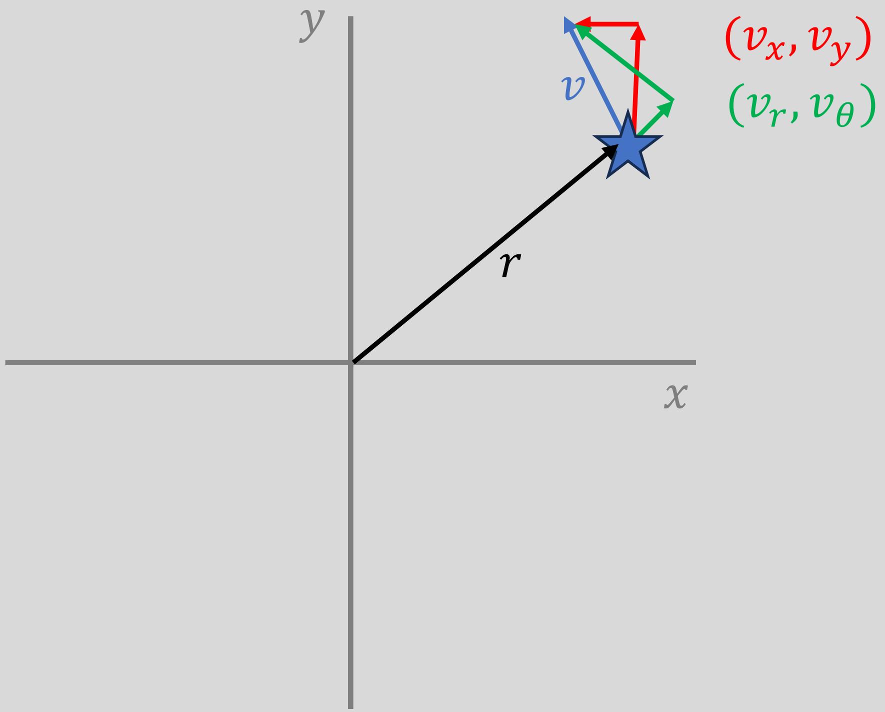
A = +15.3 ± 0.4 km/s/kpc

B = −11.9 ± 0.4 km/s/kpc

Note: additional Oort constants C and K measure non-axisymmetry.

Orbits in Axisymmetric Potentials

In non-point-mass potentials, orbits do not complete a perfect ellipse: they are not "closed". So how do we describe them?



An **integral of motion** is a quantity that is constant over an orbit:

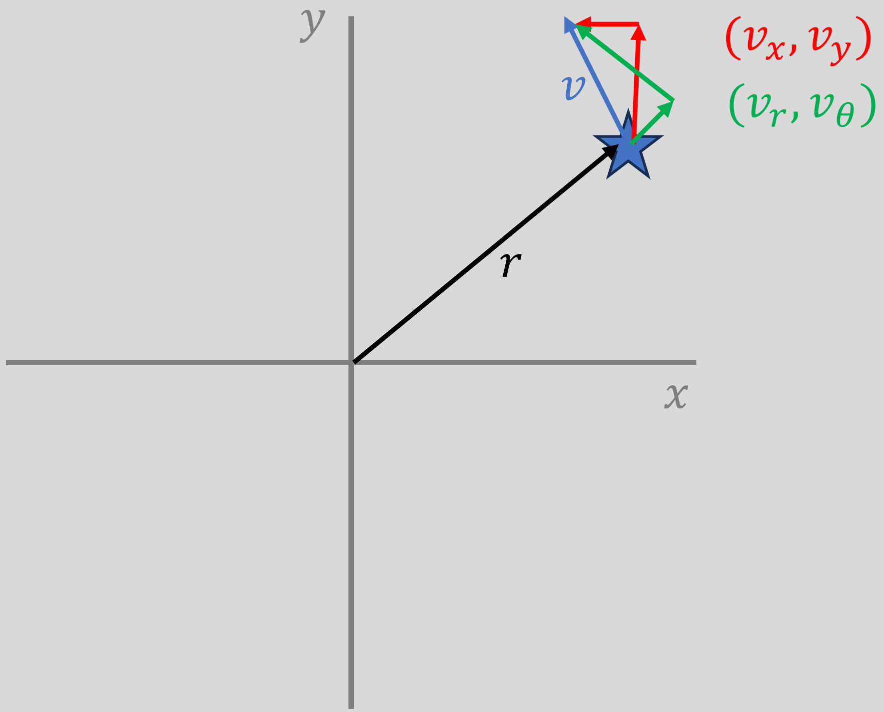
- Static Potential: Orbital Energy ($E = 0.5v^2 + \phi$)
- Spherical Potential: Total Angular Momentum ($\vec{L} = \vec{r} \otimes \vec{v}$)
- Axisymmetric Potential: L_z, the z-component of L

Look at in-plane orbital energy: $E = 0.5v^2 + \phi = 0.5\left(v_r^2 + v_\theta^2\right) + \phi$

Look at angular momentum: $|\vec{L}| = xv_y - yv_x = rv_\theta = L_z$

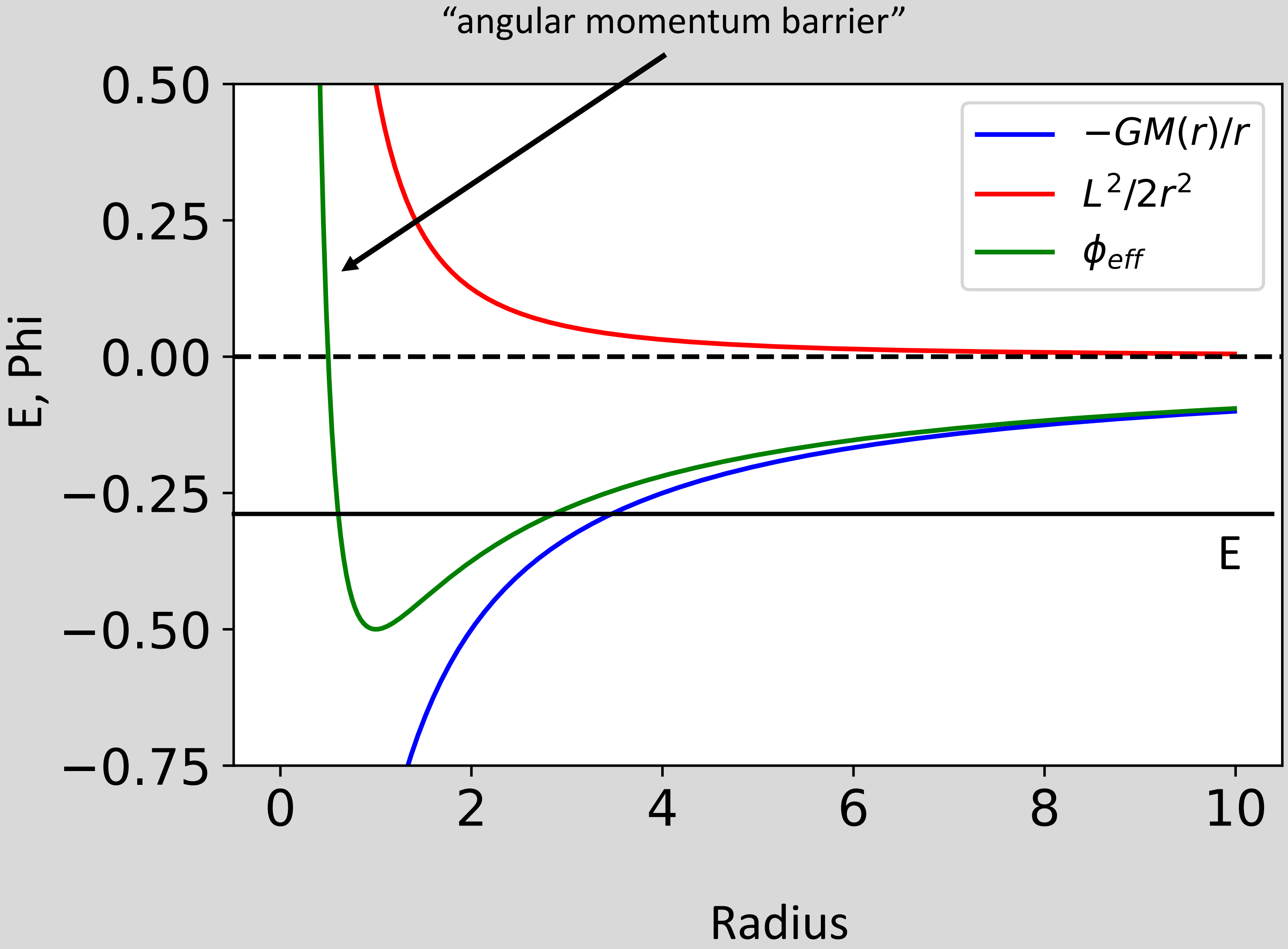
So $v_\theta = \frac{L}{r}$ and we can rewrite energy as $E = 0.5v_r^2 + 0.5\frac{L^2}{r^2} + \phi(r) \quad = 0.5v_r^2 + \phi_{eff}(r)$

where $\phi_{eff} = \phi(r) + 0.5\frac{L^2}{r^2}$ is called the **effective potential** -- a combination of the gravitational potential and the angular momentum. This turns the problem into a function of *r* alone.

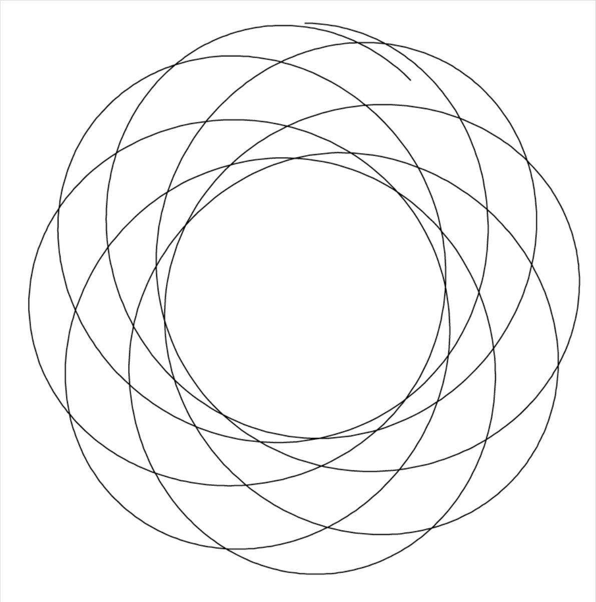


y
(v_x, v_y)
v
(v_r, v_θ)
r
x

The effective potential



At a given E, L, orbits form a rosette between percenter (r_p) and apocenter (r_a)



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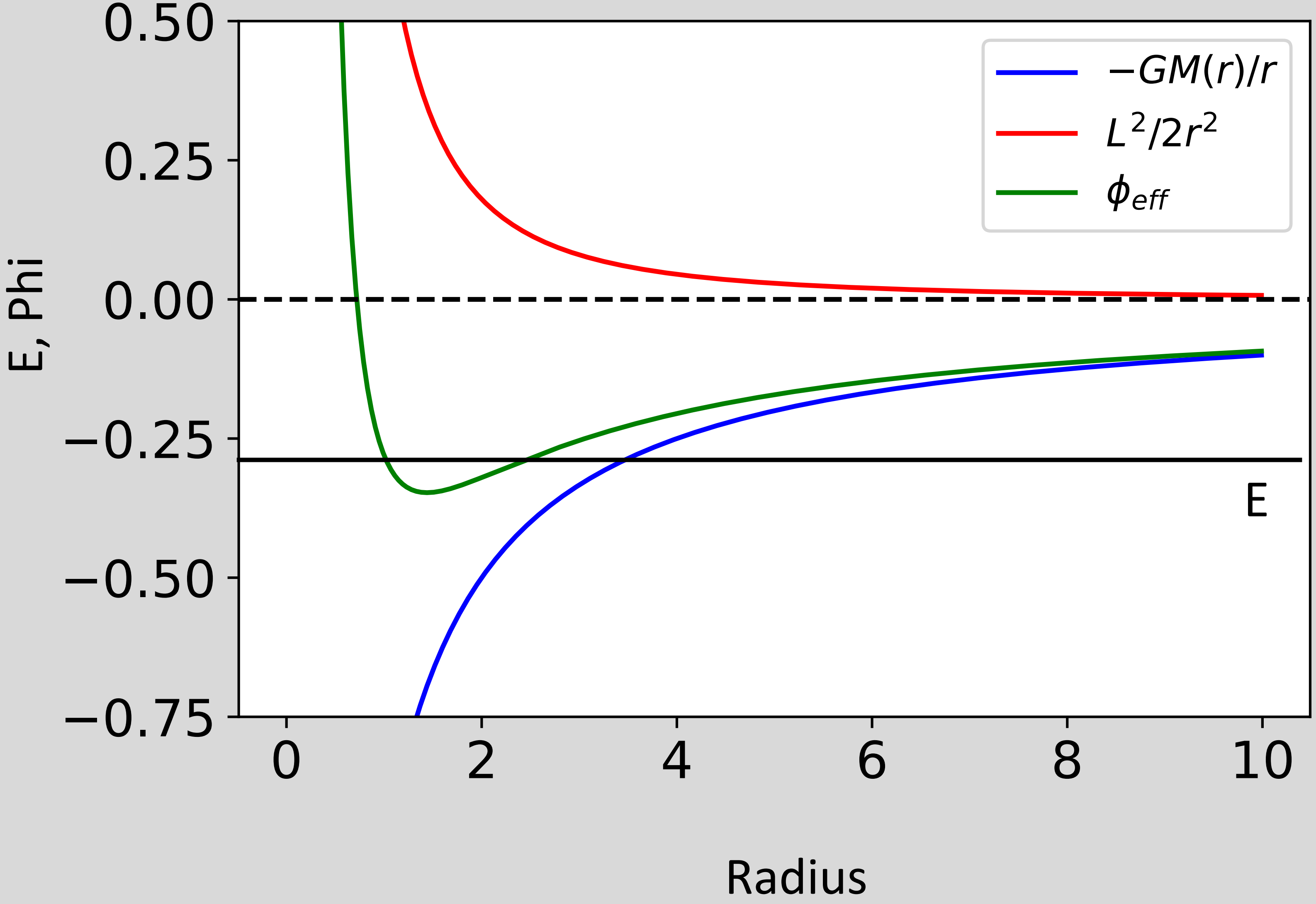

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At fixed E, increasing L reduces the range of apo and peri

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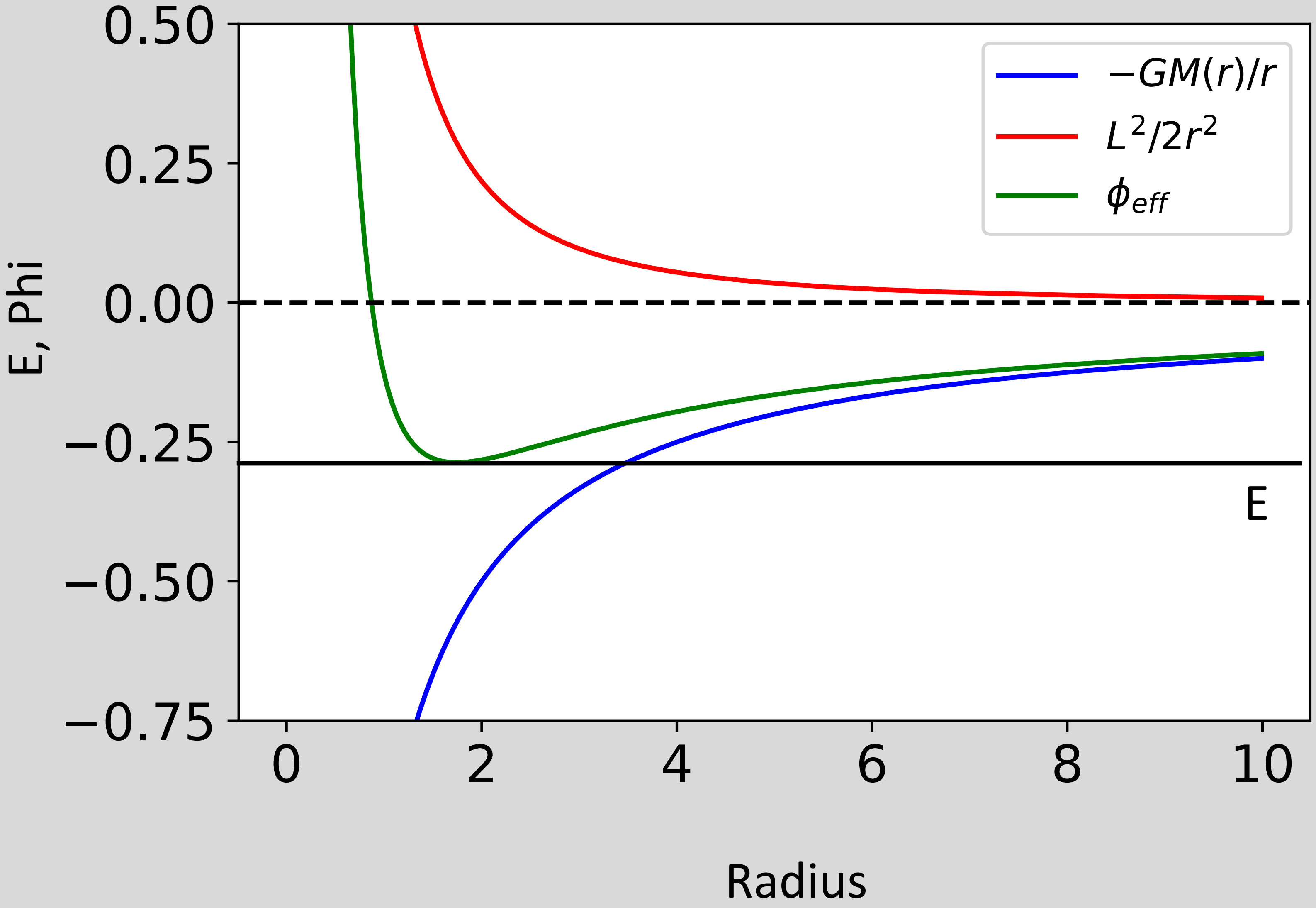
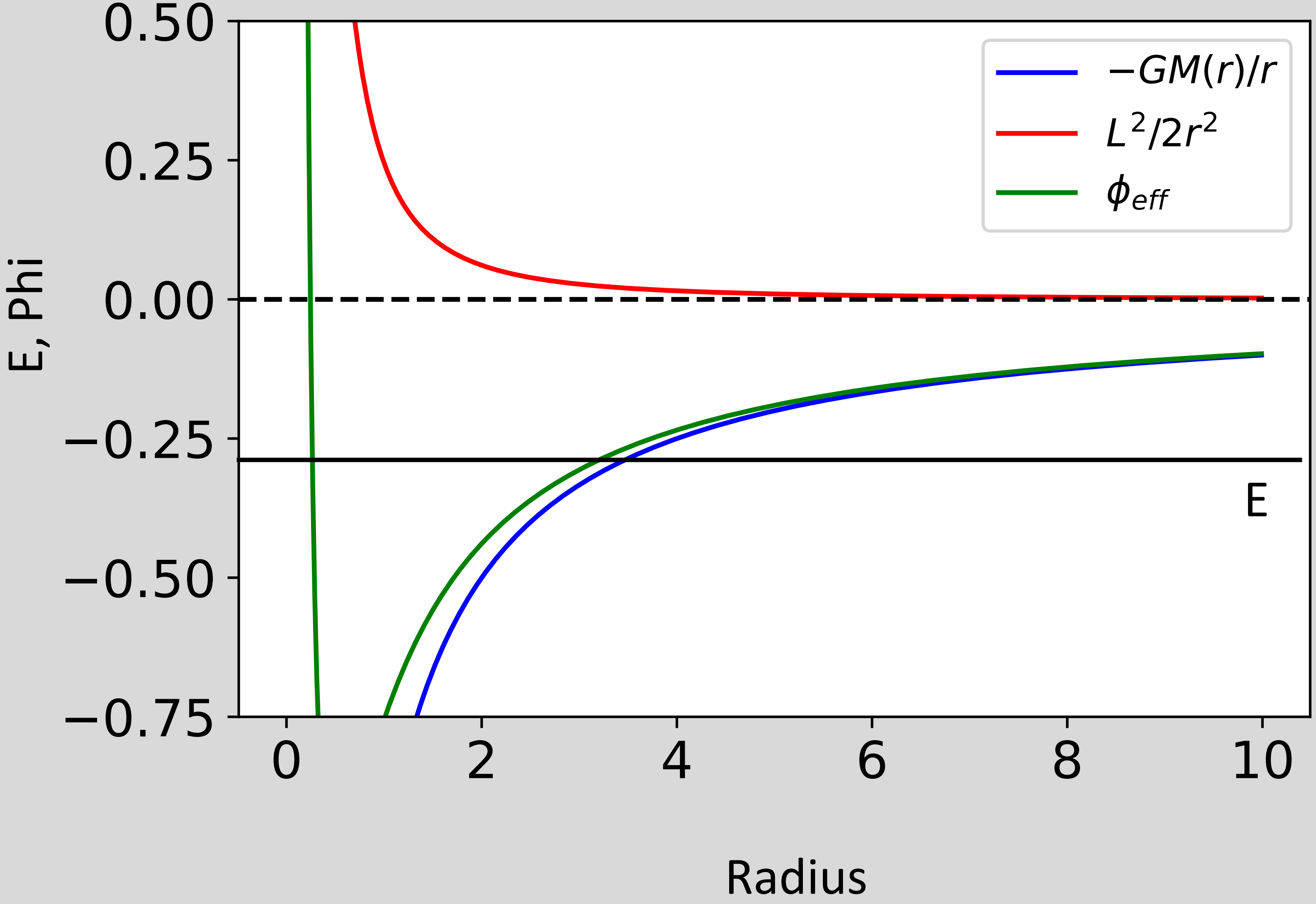


At a given E, L, orbits form a rosette between percenter (r_p) and apocenter (r_a)

At fixed E, increasing L reduces the range of apo and peri

At fixed E, highest L gives circular orbits.

The effective potential



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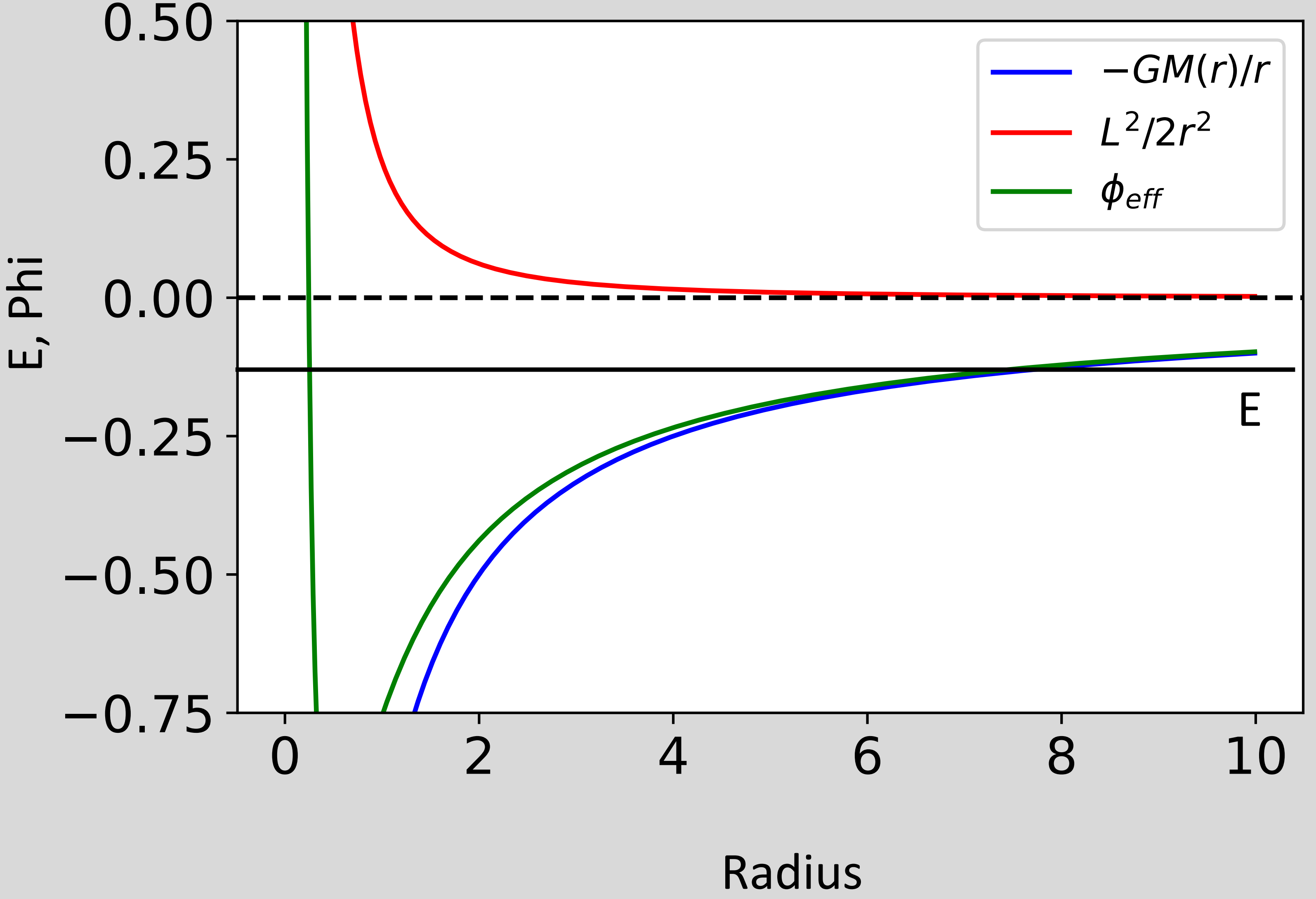
At fixed E, increasing L reduces the range of apo and peri

At fixed E, highest L gives circular orbits.

Very low L orbits can get close to the center.

The effective potential



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At fixed E, increasing L reduces the range of apo and peri

At fixed E, highest L gives circular orbits.

Very low L orbits can get close to the center.

Raising E gives more radial range to orbit.

Orbits in Axisymmetric Potentials

Remember the force acting on a star comes from the potential: $\vec{F} = m\vec{a} = -m\nabla\phi$

Separate the orbital motion into R and z motions:

$$\ddot{R} = -\frac{\partial\phi_{eff}}{\partial R} \qquad \ddot{z} = -\frac{\partial\phi_{eff}}{\partial z} \qquad \phi_{eff} = \phi(R,z) + \frac{L_z^2}{2R^2}$$

Define $x \equiv R - R_g$ where R_g is the radius of a circular orbit with angular momentum L_z

If x and z are small, we can do a Taylor expansion of the effective potential around $(x,z) = (0,0)$:

$$\phi_e = \phi_{eff}(R_g, 0) + \frac{1}{2}\left(\frac{\partial^2\phi_{eff}}{\partial R^2}\right)x^2 + \frac{1}{2}\left(\frac{\partial^2\phi_{eff}}{\partial z^2}\right)z^2 + \cdots$$

define $\kappa^2 = \left(\frac{\partial^2\phi_{eff}}{\partial R^2}\right)$ and $\nu^2 = \left(\frac{\partial^2\phi_{eff}}{\partial z^2}\right)$ and we get $\ddot{x} = -\kappa^2 x$ and $\ddot{z} = -\nu^2 z$

which are equations of harmonic oscillators with frequency κ and ν.

This is referred to as the **epicyclic approximation**, for reasons which will become clear soon….

Look at In-plane radial motion

κ is called the **epicyclic frequency**

Rewrite κ using our expression for the effective potential:

$\kappa^2 = \left(\frac{\partial^2 \phi_{eff}}{\partial R^2}\right)$ and $\phi_{eff} = \phi(R,z) + \frac{L_z^2}{2R^2}$ so we get $\kappa^2 = \left(\frac{\partial^2 \phi}{\partial R^2}\right) + \frac{3L_z^2}{R^4}$

We want to solve this in terms of orbital motion, so connect Ω , ϕ, and V_c:

$$\Omega^2 = \frac{V_c^2}{R^2} = \frac{1}{R}\frac{V_c^2}{R} = \frac{1}{R}\frac{\partial \phi}{\partial R}$$

Remember, these variables all depend on radius, they are not constants!

$$V_c = V_c(R)$$
$$\phi = \phi(R)$$
$$\Omega = \Omega(R)$$

but we can also connect it to L_z

$$\Omega^2 = \frac{V_c^2}{R^2} = \frac{V_c^2 R^2}{R^4} = \frac{L_z^2}{R^4}$$

So

$$\kappa^2 = \left(\frac{\partial^2 \phi}{\partial R^2}\right) + \frac{3L_z^2}{R^4} = \frac{\partial(R\Omega^2)}{\partial R} + \frac{3L_z^2}{R^4} = R\frac{\partial \Omega^2}{\partial R} + \Omega^2 + 3\Omega^2 = R\frac{\partial \Omega^2}{\partial R} + 4\Omega^2$$

And finally bringing in Oort Constants, $\kappa^2 = -4B(A-B)$

You will show this in HW #3

In-plane Motion: 2D oscillations

Now let's look at the 2D motion in the plane. We have $\ddot{x} = -\kappa^2 x$ which has some solution

ξ is just phase term, setting the starting point of the oscillation.

$\boxed{x = R - R_g}$

$$x(t) = X\cos(\kappa t + \xi)$$

Look at azimuthal motion. Let ψ be the angular coordinate along the orbit, so $\dot{\psi}$ is the angular velocity:

$$\dot{\psi} = \frac{L_z}{R^2} = \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2}$$

Remember, R_g is the radius of the circular orbit we are tweaking!

where I've simply substituted in $R = R_g + x$ and then done some algebra.

If $x/R_g \ll 1$, I can do another expansion to get

$$\dot{\psi} \cong \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

Now substitute x and be explicit about the derivative

$$\dot{\psi} = \frac{\partial\psi}{\partial t} = \Omega_g\left(1 - \frac{2X}{R_g}\cos(\kappa t + \xi)\right)$$

And now integrate

$$\psi(t) = \Omega_g t - \frac{2\Omega_g X}{\kappa R_g}\sin(\kappa t + \xi) + \psi_0$$

ψ_0 is just another phase term that sets the starting point of the orbit.

Finally: Epicycles

Let's put an (x,y) cartesian coordinate system centered on $(R_g, \Omega_g t + \psi_0)$.

Since $\psi(t) = \Omega_g t - \frac{2\Omega_g X}{\kappa R_g}\sin(\kappa t + \xi) + \psi_0$ we have

$$\left.\begin{aligned} x(t) &= \quad X\cos(\kappa t + \xi) \\ y(t) &= -Y\,\sin(\kappa t + \xi), \text{ where } Y \equiv \frac{2\Omega_g X}{\kappa R_g} \end{aligned}\right\}$$ *This is simply the equation of an ellipse!*

The star moves on an ellipse around R_g, as R_g moves around the galaxy on a circular orbit. The motion is described as an **epicycle** with a **guiding center** R_g! The frequency κ is called the **epicyclic frequency**.

Notes:

- The ellipse has an axis ratio of $X/Y = \kappa/(2\Omega_g)$
- For typical galactic potentials $Y > X$, so the ellipse is elongated tangentially
- Epicycles are retrograde. *Why?*
    - **Conservation of angular momentum.**
    - *When the star is further out from the guiding center it moves more slowly and lags the guiding center.*
    - *When the star is closer in, it moves more quickly and leads the guiding center.*

Epicycles around a point source

Think of Keplerian motion: $V_C \sim R^{-0.5}$, $\Omega = V_c/R \sim R^{-1.5}$,

Epicyclic frequency: $\kappa^2 = R\dfrac{\partial\Omega^2}{\partial R} + 4\Omega^2 = \mathrm{R}\dfrac{\partial(R^{-3})}{\partial R} + 4\Omega^2 = \mathrm{R}(-3R^{-4}) + 4R^{-3} = R^{-3}$

So: $\kappa \sim R^{-1.5} \sim \Omega$, and the ratio of the ellipse is $\dfrac{X}{Y} = \dfrac{\kappa}{2\Omega_g} = \dfrac{1}{2}$

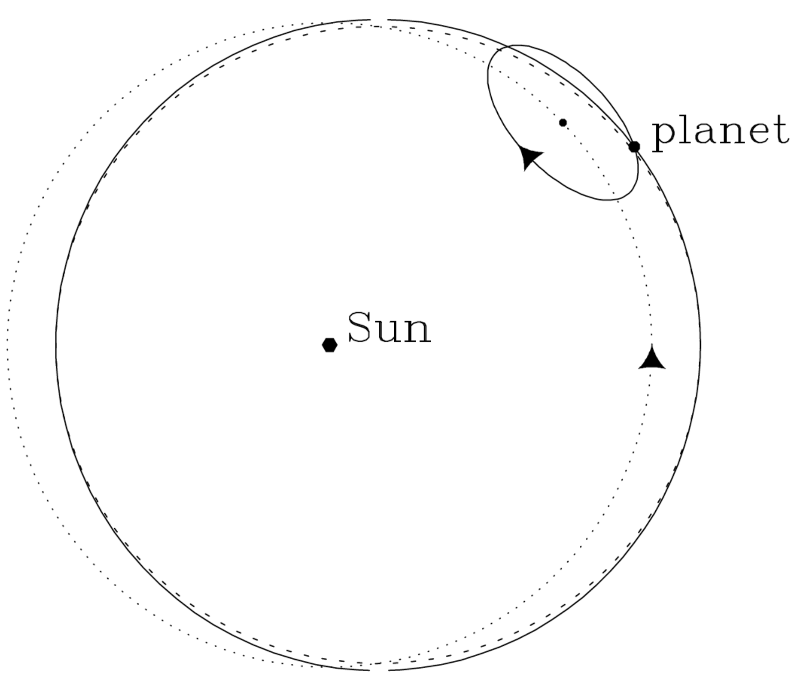
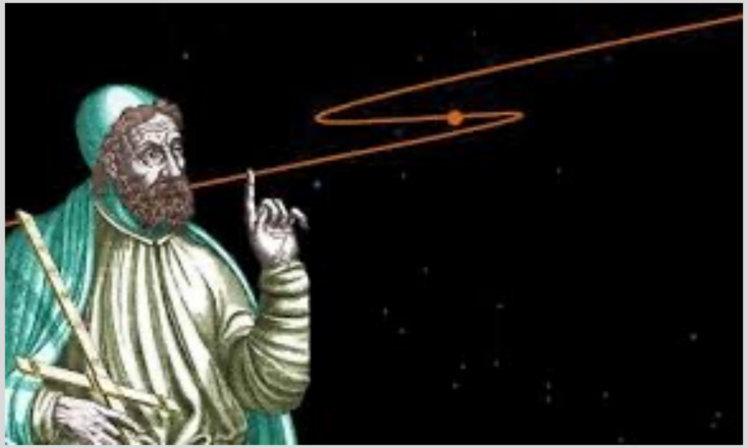


Figure 3.7 An elliptical Kepler orbit (dashed curve) is well approximated by the superposition of motion at angular frequency κ around a small ellipse with axis ratio $\frac{1}{2}$, and motion of the ellipse's center in the opposite sense at angular frequency Ω around a circle (dotted curve).

What did Ptolemy get wrong?



Clarification on the epicyclic approximation:

The epicycle:

- It is an epicyclic loop only in the rotating frame of reference.

- In any reasonable potential, the epicyclic frequency (κ) is comparable to the orbital frequency (Ω) to within a factor of a few.

- So orbits do not gyrate wildly, they just deviate slightly from circular.

Why is the epicycle retrograde compared to the guiding center mortion?

- Momentum ($L = rv_\theta$) is conserved on the orbit.
- When the star is inside R_g it has a higher angular velocity, so it moves ahead of the guiding center.
- As it moves ahead, it is moving faster than circular, so it also drifts outwards.
- As it drifts outwards it also slows down in v_θ to keep angular momentum conserved.
- As it moves beyond R_g and slows down in v_θ it lags the guiding center.
- Since it now is moving slower than circular, it starts to drift back inwards.
- and the cycle repeats, with the epicycle being retrograde compare to the guiding center motion.