

ASTR 323/423 HW #4

1. Freeman's Law (15 points)

In 1970, Ken Freeman published a study saying that disk galaxies all had a characteristic central surface brightness of $\mu_{B,0} \approx 21.7 \text{ mag/arcsec}^2$. We now know that Freeman's Law is a result of selection effects, galaxies exist at much lower central surface brightness, but can be hard to see. So let's work out some numbers.

- First, how many L_{sun} come from the central square parsec of a $\mu_{B,0} \approx 21.7 \text{ mag/arcsec}^2$ galaxy?
- If the galaxy has a total absolute magnitude of $M_B = -20.5$, what is its luminosity in solar luminosities?
- Assuming the galaxy has no bulge, show that the galaxy must have an exponential scale length of $h_R \approx 5.4 \text{ kpc}$, that $R_{25} \approx 3h_R$, and that about 80% of the galaxy's total light falls within R_{25} .
- For a low surface brightness galaxy with the same luminosity but $\mu_{B,0} \approx 24.5 \text{ mag/arcsec}^2$, show that < 10% of the light comes from within R_{25} .

Now think about a sample of spiral galaxies with luminosity $L_B = 2.5 \times 10^{10} L_{\odot}$. For these galaxies, the larger the scale length h_R , the fainter $\mu_{B,0}$ must be (right?). So, do the following:

- For the range of scale lengths $1 \text{ kpc} < h_R < 30 \text{ kpc}$, plot R_{25} (in kpc) against h_R , and R_{25} against $\mu_{B,0}$.
- Show that R_{25} is small when h_R is small, then rises to a maximum, and then declines to zero at $h_R = 24 \text{ kpc}$. Explain *physically* why this happens.
- Explain why galaxies with a central surface brightness a factor of 10 lower (in linear units; convert to surface brightness) might have been missed from Freeman's 1970 sample.

(Very small galaxies are also difficult to study: those with $R_{25} < 30''$, or 6 kpc at $d \approx 40 \text{ Mpc}$, are likely omitted from those early catalogs.)

2. Disk Stability (15 points)

Here we are going back to the disk/bulge/halo galaxy model that you built in HW #3. Using the properties of that model (or if you ran into difficulties on that problem, use the properties I gave in the solution set) let's look at the stability of the disk in that model. In class, we talked about the Toomre Q criterion, which was an estimate of *local* stability. In other words, it measures the likelihood that, at some spot in the galaxy, small regions might start to contract due to gravity.

Toomre also defined a *global* instability parameter for bar formation, which show where a galaxy might be susceptible to going unstable and forming a strong bar. He called this the X_2 stability parameter, and it is given by

$$X_2 = \frac{\kappa^2 R}{4\pi G \Sigma}$$

If $X_2 > 1$, the galaxy is stable and is not likely to form a bar, but if $X_2 < 1$, it will become unstable and can form a bar.

We are going to look at the stability parameters for two models. One will be the disk/bulge/halo galaxy model you built in HW #3, and the other will be the same disk/halo model, but with no bulge (so, you'll just set $M_{\text{bulge}} = 0$ for the second model).

For each of these models, you can calculate the Q and X_2 stability parameters as a function of radius. You'll need to know how the radial velocity dispersion (σ_r) changes with radius. To work this out, assume the galaxy is like the Milky Way: its stars have a radial velocity dispersion of $\sigma_R = 30$ km/s at the solar radius, and the velocity dispersion and disk surface density are related by $\sigma_R^2 \propto \Sigma_{disk}$. That will let you calculate σ_R as a function of radius. Also, be careful here: when using the stability formulae, remember that you'll need to convert your epicyclic frequency from km/s/kpc to km/s/pc, since if you use $G = 4.3 \times 10^{-3}$ you'll need your all your units to be in km/s, parsecs, and solar masses.

For each of your disk/halo model and your disk/bulge/halo model, make plots as a function of radius (use the same x- and y- limits for your plots that you used in HW #3) of

- The circular velocity $V(R)$
- Ω , $\Omega - \kappa/2$, and $\Omega + \kappa/2$, also marking the pattern speed of the Milky Way's spiral arms like you did in Problem 1. Note where the Lindblad resonances are in each case.
- the two stability criteria: Q and X_2 (make your y-axis limits 0 to 5 when plotting).

Finally, from what you see in these plots, discuss how the presence or absence of a central bulge is likely to affect the dynamics of the disk in terms of spiral or bar structures.

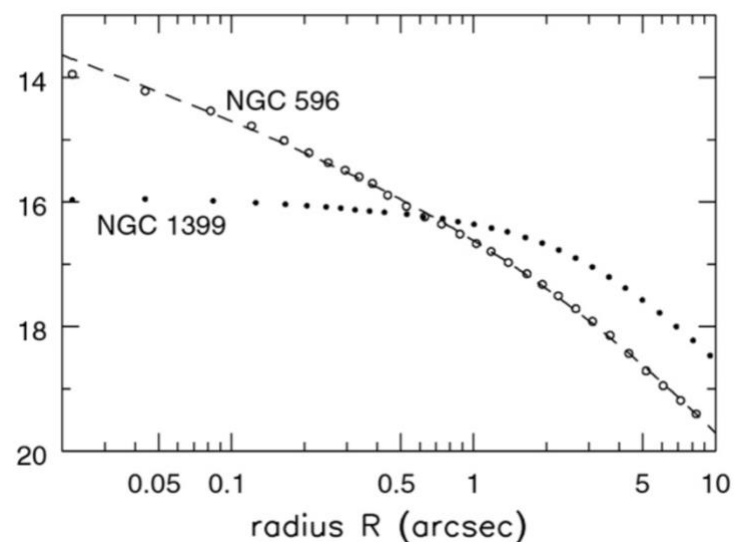
(You might also be interested in a paper by [Mihos, McGaugh, and de Blok \(1997\)](#) that discusses some of these issues as they pertain to low surface brightness galaxies as well...)

3. Elliptical Galaxy M/L ratios (15 points)

A simple estimator for the mass of a kinematically hot (dispersion dominated) system is given by $M \approx 6\sigma^2 r_c / G$, where σ is the observed velocity dispersion and r_c is the core radius of the galaxy. If we assume the galaxy has a constant luminosity density (I_0 in $L_\odot \text{ pc}^{-2}$) within its core radius, write an expression for the mass to light ratio (M/L) of the galaxy's core in terms of σ , r_c , and I_0 .

This figure shows the inner surface brightness profiles of the luminous cored elliptical galaxy NGC 1399 and the fainter, cuspy elliptical NGC 596. (The y-axis is in V mags per arcsec²). Calculate the central luminosity density of each galaxy in units of $L_\odot \text{ pc}^{-2}$. (Hint: You did something very similar in Problem #2 of HW #3; but be careful, we are working in V mags here, not B mags.)

The core radius of NGC 1399 is about 5 arcsec, or 400 pc. If the galaxy has a central velocity dispersion of $\sigma \approx 350$ km/s, show that the mass to light ratio of the galaxy's core is similar to that of globular clusters ($M/L \approx 5M_\odot/L_\odot$).



Now let's look at the globular clusters in the outskirts of NGC 1399. The size of the globular cluster system is about 9 arcminutes in radius, and the clusters show a velocity dispersion of $\sigma \approx 275$ km/s. Adopting a simple approximation which relates velocity dispersion and circular velocity as $\sigma = V_c/\sqrt{2}$, show that if NGC 1399 lies

at a distance of 20 Mpc, it must have a total mass of $M \approx 2 \times 10^{12} M_{\odot}$. If the galaxy also has an apparent magnitude of $m_V = 9.8$, work out the galaxy's total mass to light ratio.

Comparing the mass to light ratio of the core to that of the galaxy as a whole, what can you say about how the dark matter is distributed in NGC 1399?

4. Dynamical Friction (15 points)

If we adopt a density distribution for the Galaxy's dark matter halo that looks like an isothermal sphere with a circular velocity V_c :

$$\rho(r) = \frac{V_c^2}{4\pi G r^2}$$

First, work out the rotation curve for this density distribution and show that the circular velocity at all radii is constant and given by $V_c(R) = V_c$. (To do this, remember that the total mass interior to a radius r is given by $M(r) = \int_0^r 4\pi r'^2 \rho(r') dr'$.)

Now use the expression for the deceleration by dynamical friction that was given in class to show that the friction force opposing the motion of an object of mass M traveling on a circular orbit around the Galaxy is given by

$$F_{\parallel} = -\frac{GM^2}{r^2} \ln \Lambda$$

where $\ln \Lambda$ is the Coulomb logarithm. If the object spirals in slowly, we can approximate its motion as circular and write its angular momentum as $L = MrV_c$. We also know that torque ($\tau = r \otimes F$) is the rate of change of angular momentum, in other words for a circular orbit:

$$\frac{dL}{dt} = \tau = rF_{\parallel}$$

If the satellite doesn't lose mass (so $dM/dt = 0$) write down and solve a differential equation for the inspiral rate dr/dt to show that the time to sink to the center of the galaxy is

$$t_{\text{sink}} = \frac{r^2 V_c}{2GM \ln \Lambda}$$

A more complete calculation gets a value that is about twice this timescale, so let's use $2t_{\text{sink}}$ as our estimator of the sinking time. Using that estimator, do the following.

- Consider the Large Magellenic Cloud ($M \approx 2 \times 10^{10} M_{\odot}$, $r \approx 50$ kpc). Calculate an appropriate value for the Coulomb logarithm and use that to calculate the sinking time of the LMC into the center of the Milky Way.
- Now consider a $10^6 M_{\odot}$ globular cluster. Again calculate the appropriate Coulomb logarithm and then work out how close such a globular cluster has to be to the Galaxy to spiral in within 10 Gyr.

Finally, explain qualitatively how and why you think the estimate for the LMC's sinking time would change if the orbit was more elongated, or if we factored in mass loss from the LMC due to tidal stripping.

5. (ASTR 423 students only!) Flux and distance uncertainties

- Starting with the definition of the magnitude system: $m_1 - m_2 = -2.5 \log(f_1/f_2)$, use differential calculus to show that for small magnitude uncertainties, the magnitude uncertainty is related to the relative flux uncertainty by the expression $\sigma_m \approx \sigma_f/f$. In other words, show that (for example) a magnitude uncertainty of 0.1 mag is a relative flux uncertainty of about 10%.
- Starting with the definition of the distance modulus: $m - M = 5 \log(d) - 5$, use differential calculus to show that for small uncertainties in the distance modulus (σ_{m-M}), the relative uncertainty in distance is roughly half the uncertainty in distance modulus. In other words, show that $\sigma_d/d \approx 0.5\sigma_{m-M}$.

6. (ASTR 423 students only!) The Faber-Jackson Law and the Fundamental Plane (15 points)

In this problem, we are going to explore the Faber-Jackson Law and Fundamental Plane correlations for a sample of elliptical galaxies. For this exercise, use the dataset of Sloan Digital Sky Survey galaxies you downloaded in HW#2.

First, read in the galaxy dataset and use Hubble's Law to get distances for each galaxy and then also calculate their absolute g-band magnitude (M_g). Then make a plot of absolute magnitude vs $g - r$ color so that you can see the red and blue sequence. If you use this code snippet, it will produce a nice density plot that lets you see the sequences really well:

```
from matplotlib.colors import LogNorm
absmag_bins = np.linspace(-23,-12,200) # absolute mags: -23 < M_g < -12
color_bins = np.linspace(-0.5,1.5,200) # g-r color: -0.5 < g-r < 1.5
plt.hist2d(<abs-gmag>, <g-r color>, bins = (absmag_bins,color_bins), norm = LogNorm(1,100))
```

Now work out the slope and intercept values for a line that runs through the red sequence. Don't try and do a numerical fit, just estimate it by eye, overplot it, and then tweak the slope and intercept values until you think it runs through the red sequence points pretty well. That line we will refer to as the "spine" of the red sequence. If you express this line as

$$(g - r) = A_{rs} \times M_g + B_{rs}$$

then what are your values for the slope (A_{rs}) and intercept (B_{rs}) of the red sequence spine?

Now we want to pick a sample of elliptical galaxies from that dataset. We want a sample of reasonably luminous elliptical galaxies with well-measured velocities, and we want to select the sample in a way that "weeds out" any disk galaxies that might be in the dataset. So let's think about our criteria:

We know ellipticals should lie close to the red sequence spine. We also know that the light profiles of ellipticals are more concentrated (at least in the inner parts) than those of spiral galaxies. We can characterize the concentration of a galaxy as the ratio of the radius holding half the total light r_{50} to that holding 90% of the total light r_{90} , and studies have shown that ellipticals have concentrations $r_{50}/r_{90} < 0.4$.

Finally, we know that ellipticals have very low star formation rates. We can characterize something known as a "build time", which is the time it takes to build a galaxy's stellar mass given the current star formation rate: $t_{build} = \mathcal{M}_*/SFR$. Star-forming spiral galaxies have build times which are typically several billion years, while ellipticals will have very long build times: they are massive and have very little current star formation.

So we want galaxies which are:

1. reasonably luminous: absolute magnitudes brighter than -16: ($M_g < -16$)
2. having colors that lie within ± 0.1 mags of the red sequence spine:
$$|(g - r) - (A_{rs} \times M_g + B_{rs})| \leq 0.1$$
3. are centrally concentrated: $r_{50}/r_{90} < 0.4$
4. have build times which are much longer than the age of the universe: $t_{build} > 2.5 t_{universe}$
5. and have velocity dispersions in the range $50 \text{ km/s} < \sigma < 350 \text{ km/s}$

The various properties to make this selection are in the datafile. The sizes r_{50} and r_{90} (in arcsec, measured in the g-band) are in the dataset as 'petroR50_g' and 'petroR90_g'. The log of the stellar mass (\mathcal{M}_* , in units of \mathcal{M}_\odot) is given as 'lgm_tot_p50', and the log of the star formation rate (SFR , in units of $\mathcal{M}_\odot/\text{yr}$) is given as 'sfr_tot_p50'.

The easiest way to build the sample is by using a 'stacked Boolean flag' system:

```
# start with criterion 1
ellip_gals = (absmag_g < -16)
# add criterion 2
ellip_gals = np.logical_and(ellip_gals, np.abs(gr_color - (Ars*absmag_g + Brs)) < 0.1)
# and so on, stacking each criterion onto the flag.
```

After you do this, `ellip_gals` will be a Boolean array which is =1 if a galaxy meets your criteria, and is =0 if a galaxy doesn't. So you could find out how many galaxies are in your sample by saying

```
print(np.sum(ellip_gals)). And if you wanted to make a plot of two properties just for objects in your sample,
you could say:
plt.scatter(data['property1'][ellip_gals], data['property2'][ellip_gals]).
```

So how many galaxies are in your sample?

Now, make a plot of the Faber-Jackson relationship ($M_g = a \log \sigma + b$) for your sample, fit a straight line, and give the properties of the fit: slope (a), intercept (b), and scatter around the fit.

Remember that scatter is defined as the standard deviation of the points around the fitted relationship, so you want to calculate the quantity $M_g - M_{g,fit}$ ($= M_g - [a \log \sigma + b]$) for all the galaxies, and then take the standard deviation of that.

Then, fit the Fundamental Plane ($\log R_e = a \log \sigma + b \langle \mu \rangle_e + c$). For this, make sure that R_e is expressed in kiloparsecs; in other words, use the distance to convert the observed r_{50} from arcsec into physical R_e in kpc. Also you can calculate the effective surface brightness for each galaxy in mag/arcsec^2 using the relationship $\langle \mu \rangle_e = m_{g,half} + 2.5 \log(\pi r_{50}^2)$, where $m_{g,half}$ is the magnitude corresponding to *half* the total light. Doing that, **fit the Fundamental Plane and give the properties of the fit (a, b, c) and the scatter around the fit.** Then, finally to ensure the fit worked, you can **plot $\log R_e$ (on the y-axis) vs $a \log \sigma + b \langle \mu \rangle_e + c$ (on the x-axis) and then draw a line of equality ($y=x$).** If the fit worked, the line should go through the data points pretty well.

And in this case, since we are fitting $\log R_e$, to calculate the scatter, we calculate the $\log R_e - \log R_{e,fit}$ for all the galaxies and take the standard deviation of that.

Now, imagine you measure the following observed properties for an elliptical galaxy:

- apparent magnitude: $m_g = 13.6$
- angular size: $r_e = 8.2$ arcseconds
- velocity dispersion: $\sigma = 150$ km/s
- surface brightness: $\langle \mu \rangle_e = 20.2$ mag/arcsec²

Calculate the distance to the galaxy first using the Faber-Jackson relationship. Then calculate the uncertainty on that distance using the scatter in the Faber-Jackson relationship. That is, calculate the range of distances you get if you calculate them using an absolute magnitude of $M_g \pm$ FJ-scatter.

Next calculate the distance to the galaxy using the Fundamental Plane, and then also the uncertainty on that distance using the scatter in the Fundamental Plane. Meaning, again, calculate the range of distances you get if you calculate them using a physical half-light radius of $R_e \pm$ FP-scatter.

Comment on the relative accuracy of the two methods for determining distances.

Finally, if you need it, here is a code block that can do fitting of lines and planes using the astropy.modeling tools:

```
from astropy.modeling.models import custom_model
from astropy.modeling.fitting import LevMarLSQFitter

# set up two "custom_models" which are functions you want to fit. We'll
# make one for fitting a line and one for fitting a plane.

# define a line, i.e., a function which returns Y = a*X + b
@custom_model
def line(X, a = 0., b = 0.):
    return (a*X + b)

# define a plane, i.e., a function which returns Z = a*X + b*Y + c
@custom_model
def plane(X,Y, a = 0., b = 0., c = 0.):
    return (a*X + b*Y + c)

# this defines the type of fitting you are doing, in this case a least squares
# fit using a Levenberg-Marquardt algorithm
fitter = LevMarLSQFitter()

# now say you have some data xdata, ydata, zdata and you want to do some fitting

# you can fit a line to xdata and zdata by saying
model = line(a = 1.0, b = 1.0) # we give it initial guess for a and b
line_fit = fitter(model, xdata, zdata)
print(f'    slope : {line_fit.a.value:.3f}')
print(f'    intercept : {line_fit.b.value:.3f}')
# and you can ask what the fitted value of z is given xdata by saying
z_fit = line_fit(xdata)

#Or a plane to xdata, ydata, and zdata by saying
model = plane(a = 1.0, b = 1.0, c=1.0) # we give it initial guess for a, b and c
plane_fit = fitter(model, xdata, ydata, zdata)
print(f'    term1 : {plane_fit.a.value:.3f}')
print(f'    term2 : {plane_fit.b.value:.3f}')
print(f'    term3 : {plane_fit.c.value:.3f}')
# and you can ask what the fitted value of z is given xdata and ydata by saying
z_fit = plane_fit(xdata,ydata)
```