ASTR 323/423 HW #3

1. Oort Constants (15 points)

As shown in class, the Oort Constants are given by $A = -\frac{1}{2}\left(\frac{dV}{dR} - \frac{V}{R}\right)$ and $B = -\frac{1}{2}\left(\frac{dV}{dR} + \frac{V}{R}\right)$. From these definitions, and our expressions for the orbital angular speed (Ω) and epicyclic frequency (κ) derive the following dynamical relationships involving the Oort Constants:

i. Angular Frequency: $\Omega = A - B$

ii. Logarithmic gradient of Angular Frequency: $\frac{d \ln \Omega}{d \ln R} = -\frac{2A}{A-B}$

iii. Logarithmic gradient of the Rotation Curve: $\frac{d \ln \mathrm{V}}{d \ln R} = -\frac{A+B}{A-B}$

iv. Logarithmic gradient of the radial acceleration: $\frac{d \ln \Omega^2 R}{d \ln R} = 1 - \frac{4A}{A-B}$

v. Epicyclic Frequency: $\kappa^2 = -4B(A-B)$

Hint: In doing some of these derivations, it will be helpful to remember that $d(\ln x) = dx/x$.

For item (iv), also explain why $\Omega^2 R$ is proportional to the radial acceleration.

Now, using the values for the Oort Constants given in class ($A = +15.3$ km/s/kpc, $B = -11.9$ km/s/kpc), work out the following:

- The frequency of circular orbits (Ω) in km/s/kpc
- The circular orbital period, in millions of years
- The epicyclic frequency (κ) in km/s/kpc
- The ratio of the epicyclic and orbital frequencies (κ/Ω)
- Show whether the rotation curve in the solar neighborhood is rising, falling, or constant with radius.

2. The Properties of the Spiral Galaxy M101 (15 points)

There is a datafile waiting for you at
http://burro.case.edu/Academics/Astr323/HW/HW3/M101phot_unbinned_azi.dat

That file has photometry for M101 (taken from Mihos+12): surface brightness in B and V (in mag/arcsec^2) as a function of radius (in arcminutes). Plot the B-band surface brightness as a function of radius (please scale y-axis so that bright is towards the top), and fit a straight line to the profile. From the parameters of your fit, work out the following things:

i. The central surface brightness (μ_0, in mag/arcsec^2)
ii. The radial scale length of the disk (h, in arcmin)
iii. R_{25}, the radius of the $\mu_B = 25$ mag/arcsec^2 isophote (in arcmin)
iv. The total apparent B magnitude[1] of the galaxy
v. The B-band central luminosity density[2] of the galaxy ($I_{0,B}$, in L_{sun}/pc^2)

On your plot, mark the level of the night sky brightness ($\mu_B \approx 22.5$ mag/arcsec^2), and also mark where the surface brightness of the galaxy drops below 1% of that night sky brightness.
Now adopt a distance of d=6.9 Mpc, and work out the following:

vi. The radial scale length in kpc
vii. The absolute B magnitude of the galaxy
viii. The B-band luminosity of the galaxy (in solar units)

Finally, use the Tully-Fisher relationship to estimate M101's circular velocity. In the 1910s, Adrian van Maanen claimed to have detected M101's rotation by observing the proper motion of stars in its outer disk. How long would it take a star at R_{25} to move 1 arcsecond due to its orbital motion? If van Maanen's measurement of proper motion was correct (roughly 1" motion over 10 years), how fast would M101 have to be rotating?

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[1] For this, remember in HW #2 that you worked out the total luminosity of an exponential disk was given by $L_{tot} = 2\pi I_0 h^2$, where I_0 was the central luminosity density (in $L_\odot\, pc^{-2}$) and h was the scale length (in pc). But the same relationship holds for the observed quantities: $f_{tot} = 2\pi f_0 h^2$, where f_{tot} is the total flux, f_0 is the central flux density (flux per square arcseconds) and h is now the scale length in arcseconds). And since magnitude is simply given by $m_{tot} = -2.5\log f_{tot} + C$, we can rewrite this as

$$\begin{aligned} m_{tot} &= -2.5\log(2\pi f_0 h^2) + C \\ &= -2.5\log(2\pi h^2) - 2.5\log(f_0) + C \\ &= -2.5\log(2\pi h^2) + \mu_0 \end{aligned}$$

Just remember that since you have measured μ_0 in mag/arcsec^2, you want h in arcseconds as well!

[2] For this, remember the connection between μ and I that you worked out back in ASTR222:

$$\mu_B = -2.5\log I_B + 27.07$$

3. Milky Way Rotation Curve (15 points)

In Problem #1, we used the measured values of the Oort Constants to work out kinematic properties of the Milky Way's disk near the location of the Sun. Now we are going to switch it around and use the Milky Way rotation curve to work out how the kinematic properties of the disk change with radius.

A smooth model of the Milky Way rotation curve (courtesy Prof McGaugh!) can be found at
http://burro.case.edu/Academics/Astr323/HW/HW3/MWvrot_BovyRix_RAR.dat
That data file gives radius in kpc and rotation speed in km/s.

From that data, make the following plots (only plot from R=0 to R=20 kpc):

- the rotation curve, V(R).
- a plot of the Oort A and B constants and the epicyclic frequency κ as a function of radius. Your units of these values will all be km/s/kpc, and set your plot limits to run from -150 to $+150$ km/s/kpc.
- a plot of Ω, $\Omega - \kappa/2$, and $\Omega + \kappa/2$ as a function of radius, setting your plot limits to run from 0 to $+100$ km/s/kpc. On this plot, also mark a horizontal line showing the pattern speed of the Milky Way spiral arms. A recent estimate of this pattern speed from Gaia DR2 data is $\Omega_p = 28.2 \pm 2.1$ km/s/kpc (Dias+ 19).

Then also calculate the following values:

- The epicyclic frequency and orbital angular frequency measured at the solar circle (R_{sun}=8.2 kpc).
- The Oort A and B values measured at the solar circle. Compare these to the values given in class; they should agree pretty well.
- The locations of the inner and outer Lindblad resonances, as well as the co-rotation radius — the radius where the orbital frequency and the pattern speed are the same. How far away is the Sun from the co-rotation radius?

Coding tip #1: To differentiate the rotation curve and get dV/dR, its easiest just to do this numerically, like this:

```
dVdR=np.diff(MWdata['V'])/np.diff(MWdata['R'])
V=0.5*(MWdata['V'][1:]+MWdata['V'][:-1])
R=0.5*(MWdata['R'][1:]+MWdata['R'][:-1])
```

The first line differentiates the rotation curve by taking the differences at each step, while the second two lines give you V and R averaged between each step. That. way dVdR, V, and R are all arrays of the same length, which makes them easy to plot.

Coding tip #2: If you need to get a value measured at a particular radius, like the solar circle (Rsun), you can do something like this:

```
Rsun_idx=np.argmin(np.abs(R-Rsun))
print(V[Rsun_idx])
```

The first line works out which index in the R array is closest the the solar circle, then the second line prints the velocity corresponding to that radius.

4. Spherical Density profiles (10 points)

Adopt a spherical mass density profile for the stars in a galaxy's bulge that is given by

$$\rho_{bulge}(r) = \frac{M_{bulge}}{2\pi}\frac{a}{r}\frac{1}{(r+a)^3}$$

where M_{bulge} is the total mass of the bulge and a is a characteristic scale radius. From this, show that[3] the enclosed mass (the total amount of mass inside a given radius) is given by

$$M(<r) = M_{bulge}\frac{r^2}{(r+a)^2}$$

and then from that work out the circular velocity as a function of radius $V_c(r)$.

Finally, work out an analytic expression for the effective radius of the bulge, which is the radius that contains half the mass. In other words, you want to work out the radius at which $M(<r) = M_{bulge}/2$.

5. Galaxy Mass Models (15 points)

Let's build some mass models which reproduce a Milky Way like rotation curve. These models will consist of a disk, a bulge, and a dark matter halo.

To work out the rotation speed due to the disk (only), use an exponential disk model, which gives a rotation speed which looks like this:

$$V_{c,disk}^2(R) = 4\pi G\Sigma_0 h_R y^2\big(I_0(y)K_0(y) - I_1(y)K_1(y)\big)$$

where Σ_0 is the mass density at R=0, h_R is the radial scale length, I_0, K_0, I_1, K_1 are Bessel functions[4], and $y = R/(2h_R)$. If the scale length of the disk is $h_R = 2.5$ kpc and the mass density of the disk at the solar circle is $\approx 50\ M_\odot\ pc^{-2}$, calculate both the central mass density (Σ_0) and total mass of the disk.

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[3] for this problem the following integral will be useful:

$$\int \frac{x}{(x+a)^3}dx = -\frac{(a+2x)}{2(a+x)^2}$$

[4] in python, just access these by saying: `from scipy.special import i0,k0,i1,k1`

For the Milky Way's bulge, use the density model for bulges that you worked out in Problem 3. To work out the right values for total bulge mass (M_{bulge}) and bulge scale radius (a), adopt a bulge-to-disk mass ratio of 1:3 and a bulge effective radius of 2.4 kpc, and use that information to work out M_{bulge} and a.

You can then insert those quantities into the expression you worked out in Problem 3 for the circular velocity due to the bulge (only).

Finally, to work out the rotation speed due to the halo (only), we will adopt the isothermal halo model, which has a density distribution given by

$$\rho_{halo}(r) = \frac{\rho_0}{1 + (r/r_c)^2}$$

where ρ_0 is the central density and r_c is the core radius. This gives a rotation curve of

$$V_{c,halo}^2(r) = 4\pi G \rho_0 r_c^2 \left(1 - \frac{r_c}{r} \arctan\left(\frac{r}{r_c}\right)\right)$$

Now, to calculate the ***total*** rotation curve for your model, remember that velocities add in quadrature, so the total rotation curve for you model will be given by

$$V_{c,tot} = \sqrt{V_{c,disk}^2 + V_{c,bulge}^2 + V_{c,halo}^2}$$

Given the values for your disk and bulge models, calculate and plot the observed Milky Way rotation curve (the dataset from problem 3) plus the "disk-bulge only" model rotation curve (so set $\rho_0 = 0$ for the halo). You should be able to see the problem: the rotation speed of the model simply doesn't match the data.

Then start building a composite model by adopting halo parameters $\rho_0 = 1.0\ M_{\odot}\ pc^{-3}$, $r_c = 300\ pc$. (Note: these are absolutely incorrect values, but they'll get you started.) Have your code plot five curves on ***one single plot***:

1. the observed rotation curve
2. the "disk only" model rotation curve
3. the "bulge only" model rotation curve
4. the "halo only" model rotation curve
5. the total "disk+bulge+halo" model rotation curve.

You'll probably want to color code or line-style your lines in a way that is intuitive to you, and then also make sure you have a legend plotted, or else it will get very confusing!
Once your code is working, fiddle around with the halo parameters (ρ_0, r_c) until the total rotation curve looks like a reasonable match to the observed rotation curve. No need to do any fancy χ^2 fitting, you just want to get the important things right: the circular speed at the solar radius ought to be ≈ 220 km/s, the rotation curve shouldn't rise too fast or too slow, and it ought to be falling ever so gently between 5 and 15 kpc, but be pretty flat by the time you get to 20 kpc. You should be able to get the curves to agree pretty well over the whole radial range.

Note that increasing r_c will make the halo rotation curve rise more slowly (and vice versa), but if you change r_c you'll need to adjust ρ_0 as well, since the two parameters are coupled. If you want to raise the rotation curve (at fixed r_c), raise the central density and that will give you more mass in the halo. (By plotting the rotation curve components separately, it's easier to see how you want to adjust the parameters of the halo to give a decent match.)

Figure out your "best fit" match, tell me your reasoning (describe how you iterated in on a solution), show me the best fit plot, and give me your best fit values for ρ_0 and r_c. Discuss the robustness of your fit as well as any degeneracies. In other words, if you take your best fit parameters and change one of them a bit, can you get back to a good fit by adjusting the other one?